Complexity of Manipulating Sequential Allocation

Haris Aziz
Data61, CSIRO and UNSW
Sydney, Australia
haris.aziz@data61.csiro.au

Sylvain Bouveret
LIG - Grenoble INP
France
sylvain.bouveret@imag.fr

Jérôme Lang
Université Paris-Dauphine, PSL Research University
CNRS, LAMSADE, Paris, France
lang@lamsade.dauphine.fr

Simon Mackenzie
Carnegie Mellon University,
Pittsburg, USA
simonm@andrew.cmu.edu

Abstract

Sequential allocation is a simple allocation mechanism in which agents are given pre-specified turns in which they take one item among those that are still available. It has long been known that sequential allocation is not strategyproof. This raises the question of the complexity of computing a preference report that yields a higher utility than the truthful preference. We show that the problem is NP-complete for one manipulating agent with additive utilities and several non-manipulating agents. In doing so, we correct a wrong claim made in a previous paper. We then give two additional results. First, we present a polynomial-time algorithm for optimal manipulation when the manipulator has additive binary utilities. Second, we consider a stronger notion of manipulation whereby the untruthful outcome yields more utility than the truthful outcome for all utilities consistent with the ordinal preferences; for this notion, we show that a manipulation, if any, can be computed in polynomial time.

Introduction

A simple but popular mechanism to allocate indivisible items is *sequential allocation* (Aziz, Walsh, and Xia 2015; Bouveret and Lang 2011; Brams and Straffin 1979; Brams and Taylor 1996; Kalinowski, Narodytska, and Walsh 2013; Kalinowski et al. 2013; Kohler and Chandrasekaran 1971; Levine and Stange 2012; Tominaga, Todo, and Yokoo 2016). In sequential allocation, a sequence specifies the turns of the agents. For example, for sequence 1212, agents 1 and 2 alternate with agent 1 taking the first turn. Agents report their preferences by expressing a linear order over items, and are allocated items based on their reported preferences as follows. They get turns according to the sequence, and when their turn comes, they are given the most preferred item (according to their reported order) that has not yet been allocated. Sequential allocation is an ordinal mechanism since the outcome only depends on the ordinal preferences of agents over items. Nevertheless, it is a standard assumption in the literature that agents have underlying additive utilities for the items.[1] It has long been known that sequential allocation is not strategyproof in particular when agents do not have consecutive turns. This motivates the natural problem of computing best responses (also referred to as manipulations). Kohler and Chandrasekaran (1971) presented a polynomial-time algorithm to compute the optimal manipulation of an agent when there are two agents and the sequence is alternating (121212..). Bouveret and Lang (2011) initiated further work on manipulation of sequential allocation and showed that (1) it can be checked in polynomial time whether an agent can be allocated a given subset of items; then, they show in a later work (Bouveret and Lang 2014) that (2) an optimal manipulation for two agents (one manipulator and one non-manipulator) can be found in polynomial time for any sequence. Now, they also claim (page 142, left column, lines 5-7) that by putting together (1) and (2), a best response can be computed in polynomial time for one manipulator against *several* non-manipulators, and that each best response results in the same allocation for the manipulator. As we will show, this conclusion is wrong.

Results We focus on computing best responses (or manipulations) under sequential allocation. We first show that the algorithm given by Bouveret and Lang (2014) for computing a best response against one non-manipulator does not extend to several non-manipulators. We then show that the problem of computing a best response is in fact NP-complete. The result has some interesting consequences since many allocation rules are based on sequential allocation. We also give two additional results. We first present a polynomial-time algorithm for optimal manipulation when the manipulator has binary utilities. We then consider a stronger notion of manipulation whereby the untruthful outcome yields more utility than the truthful outcome for all utilities consistent with the ordinal preferences. For this notion, we give a polynomial-time algorithm for computing a manipulation.

Preliminaries

We consider a set of agents $N = \{1, \dots, n\}$ and a set of items $O = \{o_1, \dots, o_m\}$. Agent 1 is the *manipulator* and agents 2 to n the *non-manipulators*. The preference profile $\succ = (\succ_1, \dots, \succ_n)$ specifies for each agent i his complete,

[1] We view here sequential allocation as a centralized mechanism; it can also be seen as a decentralized mechanism, where instead of reporting their preference, agents pick a remaining item each time their turn come. Our results hold for both views but are easier to state for the centralized view.

strict, and transitive preference $\succ_i$ over O. A preference relation $\succ_i$ is usually denoted by a list, such as $i : a, b, c, d$ for $a \succ_i b \succ_i c \succ_i d$. Moreover, for agent 1 we are also given a set of positive values $\{u_1(o) | o \in O\}$, where for each o, $u_1(o) > 0$ is 1's cardinal evaluation of the value of item o. These values are consistent with $\succ_1$, that is, $u_1(o) > u_1(o')$ iff $o \succ_1 o'$. We assume that agent 1's preference over sets of items are represented by the additive utility function $u_i(O') = \sum_{o \in O'} u_i(o)$ for all $O' \subseteq O$. We say that u_1 is *lexicographic* if for all $o \in O$, $u_1(o) > \sum_{o' \prec_i o} u_1(o')$. We abbreviate $u_1(S) \geq u_1(T)$ into $S \succeq_1 T$. Finally, an assignment p is a function from O to N.

Complexity of Manipulation

We first show that the best response algorithm of Bouveret and Lang (2014) does not work for $n \geq 3$, and that several optimal manipulations may result in different allocations.

Example 1. *Let the sequence be 1231, and consider the following profile $\succ$.*

$$\succ_1: \quad a, b, c, d \qquad \succ_2: \quad c, d, a, b \qquad \succ_3: \quad a, b, c, d$$

According to (Bouveret and Lang 2014), the best response is one in which agent 1 gets $\{a, d\}$ which can even be achieved by the truthful report. Now, if agent 1 misreports

$$\succ_1': \quad c, b, a, d$$

then he gets $\{b, c\}$. Which one of $\{a, d\}$ and $\{b, c\}$ is better depends on u_1 and not just on $\succ_1$. For example, if $u_1(a) = 5$, $u_1(b) = 4$, $u_1(c) = 3$ and $u_1(d) = 1$, then $\{b, c\} \succ_1 \{a, d\}$ and the best achievable allocation for 1 is $\{b, c\}$. On the other hand, if $u_1(a) = 4$, $u_1(b) = 3$, $u_2(c) = 2$ and $u_1(d) = 1$, then 1 is indifferent between $\{a, d\}$ and $\{b, c\}$.

Thus, for more than one non-manipulator the algorithm of Bouveret and Lang (2014) does not necessarily compute a best response,[2] and the complexity of computing a best response is an open problem. We are now going to show that the problem is NP-hard. The reduction involves a similar high-level idea as that of the result of Aziz et al. (2015a) that manipulating the probabilistic serial (PS) mechanism is NP-hard. However, the reduction requires new gadgets. Also note that the NP-hardness result for the PS mechanism does not directly imply a similar result for sequential allocation. Similarly, the NP-hardness of the manipulation of sequential allocation does not imply the NP-hardness of the manipulation of the PS mechanism.

Theorem 2. *Computing a best response for the sequential allocation mechanism is NP-complete.*

Proof. We prove that the following problem (BEST RESPONSE) is NP-complete: given an assignment setting, a utility function $u_1 : O \to \mathbb{N}$ for the manipulator, and a target utility T, can the manipulator specify preferences such that the utility for his allocation under the sequential allocation rule is at least T? The problem BEST RESPONSE is clearly in NP, since the outcome with respect to the reported preference can be computed by simulating sequential allocation.

We show hardness by reduction from a restricted NP-complete version of 3SAT where each literal appears exactly twice in the set of clauses. The problem remains NP-complete (Berman, Karpinski, and Scott 2003). Given such a 3SAT instance $F = (X, C)$ where $X = \{x_1, \ldots, x_{|X|}\}$ is the set of variables and C the set of clauses, we build an instance of BEST RESPONSE where the manipulator can obtain utility $\geq T$ if and only if C is satisfiable. We denote by $L = \{x_1, \neg x_1, \ldots, x_{|X|}, \neg x_{|X|}\}$ the set of literals.

We define the set of agents as

$$N = \{1\} \cup \{a_{l_i}^1, a_{l_i}^2 : l_i \in L\}$$

with agent 1 as the manipulator, and the set of items as

$$O = \{o_c^1, o_c^2, o_c^3 : c \in C\} \cup \{o_{l_i}^1, o_{l_i}^2, h_{l_i}^1, h_{l_i}^2, h_{l_i}^3, d_{l_i}^{11}, d_{l_i}^{12}, d_{l_i}^{21}, d_{l_i}^{22} : l_i \in L\}$$

with

- $O(c) = \{o_c^1, o_c^2, o_c^3\}$: *clause items* associated with clause c
- $O(l_i) = \{o_{l_i}^1, o_{l_i}^2\}$: *choice items* associated with literal l_i
- $H(l_i) = \{h_{l_i}^1, h_{l_i}^2, h_{l_i}^3\}$: *consistency items* associated with l_i
- $D(l_i) = \{d_{l_i}^{11}, d_{l_i}^{12}, d_{l_i}^{21}, d_{l_i}^{22}\}$: *dummy items* associated with l_i

We view the sequential allocation process as follows. The agents' preferences are built in a way such that the agents will first go through $|X|$ *choice rounds* corresponding to variables $x_1, \ldots, x_{|X|}$, then $|C|$ *clause rounds* corresponding to $c_1, \ldots, c_{|C|}$, and then one final *collection round.*

High-level Idea The picking sequence is composed of successive *rounds*. Each round is associated with a subset of items that are valued by agent 1 much higher than items associated with later rounds. If he wants to reach the target utility T, then in each round, agent 1 must focus only on the items relevant to that round; those he will not get during the corresponding round will be taken by other agents and it will not be possible for him to take them in later rounds. In each round, agent 1 makes a choice between the items corresponding to a variable and to its negation. There is a negligible difference between the utility of the items corresponding to the variable and those corresponding its negation (for example o_x^1 and $o_{\neg x}^1$). If he chooses the items in a 'correct' way, then he will get a most preferred item corresponding to each of the clauses: more precisely, agent 1, in any case, will get one item $h_{x_i}^j$ and one item $h_{\neg x_i}^k$ for each variable x_i; if he chooses in a 'correct' way, then this pair of items $\{h_{x_i}^j, h_{\neg x_i}^k\}$ will give him a sufficiently high utility, otherwise there will be a utility loss which he will never be able to compensate. We show below that agent 1 reaches utility T if and only if these two conditions are satisfied:

[2] The reason is that when they translate the n-agent problem P of deciding whether agent 1 can obtain a set of objects S into a 2-agent problem P' (Bouveret and Lang 2011), the preference relation of agent 2 in P' problem depends on S, and in the best response algorithm for a 2-agent problem of Bouveret and Lang (2014), S is not fixed.

1. he chooses the choice items in a consistent way, that is, at each choice round i he chooses either $o^1_{x_i}$ and $o^2_{x_i}$ (which we view as assigning x_i to false), or $o^1_{\neg x_i}$ and $o^2_{\neg x_i}$ (which we view as assigning x_i to true).
2. he manages to get his most preferred clause item for each clause c; this is possible only if none of the agents corresponding to the negation of the literals in c gets this clause item before him.

Utility function of agent 1 Agent 1 highly prefers choice items relevant to variable x_i to items relevant to variable x_j, $j > i$; for a given variable he highly prefers its relevant choice items to its relevant consistency items. These items are highly preferred to clause items, which in turn are highly preferred to dummy items:

$$O(x_1), O(\neg x_1) \gg_1 H(x_1), H(\neg x_1) \gg_1 \ldots \gg_1 O(x_{|X|}), O(\neg x_{|X|}) \gg_1 H(x_{|X|}), H(\neg x_{|X|}) \gg_1 o^1_{c_1} \gg_1 o^1_{c_2} \gg_1 \ldots \gg_1 o^1_{c_{|C|}} \gg_1 \text{other items}$$

His utilities for items in $O(x_i)$ and $H(x_i)$ are as follows:

- $u_1(o^1_{x_i}) = u_1(o^1_{\neg x_i}) + \epsilon \gg u_1(o^2_{x_i}) = u_1(o^2_{\neg x_i}) + \epsilon$, where ϵ is a negligible quantity.
- $h^1_{\neg x_i} \succ_1 h^2_{\neg x_i} \succ_1 h^3_{\neg x_i} \succ_1 h^1_{x_i} \succ_1 h^2_{x_i} \succ_1 h^3_{x_i}$.
- $u_1(h^2_{x_i}) + u_1(h^2_{\neg x_i}) < u_1(h^1_{\neg x_i}) + u_1(h^3_{x_i}) = u_1(h^1_{x_i}) + u_1(h^3_{\neg x_i})$.
- agent 1 reaches utility T if and only if he gets two items corresponding to a variable, at least one top choice consistency item (h^1_x or $h^1_{\neg x}$) in each round and his target clause item in each clause round.

Choice Round The choice round is composed of a series of rounds, one for each variable x_i; each of them is itself composed of two subrounds; in each of them, agent 1 has to choose a choice item. The sub-sequence in choice round corresponding to variable x_i is as follows:

$$1, a^1_{\neg x_i}, a^2_{\neg x_i}, a^1_{x_i}, a^2_{x_i}, 1, a^1_{\neg x_i}, a^2_{\neg x_i}, a^1_{x_i}, a^2_{x_i}, a^1_{\neg x_i}, a^2_{\neg x_i}, 1, a^1_{x_i}, a^2_{x_i}, 1$$

The ordinal preferences relevant for the choice round corresponding to variable x_i are as follows. Recall that agent 1's preferences are

$$1: \quad o^1_{x_i}, o^1_{\neg x_i}, o^2_{x_i}, o^2_{\neg x_i}, h^1_{\neg x_i}, h^2_{\neg x_i}, h^3_{\neg x_i}, h^1_{x_i}, h^2_{x_i}, h^3_{x_i}$$

Then, each literal agent prefers the items corresponding to the negation of the literal:

$$a^1_{\neg x_i}: \quad o^1_{x_i}, d^{11}_{x_i}, d^{12}_{x_i}, o^2_{x_i}, h^1_{x_i}, h^2_{x_i}, h^3_{x_i}$$
$$a^2_{\neg x_i}: \quad d^{21}_{x_i}, o^1_{x_i}, o^2_{x_i}, d^{22}_{x_i}, h^1_{x_i}, h^2_{x_i}, h^3_{x_i}$$
$$a^1_{x_i}: \quad o^1_{\neg x_i}, d^{11}_{\neg x_i}, h^1_{\neg x_i}, o^2_{\neg x_i}, h^2_{\neg x_i}, h^3_{\neg x_i}, d^{12}_{\neg x_i}$$
$$a^2_{x_i}: \quad d^{21}_{\neg x_i}, o^1_{\neg x_i}, o^2_{\neg x_i}, h^1_{\neg x_i}, h^2_{\neg x_i}, h^3_{\neg x_i}, d^{22}_{\neg x_i}$$

Items that do not appear in these lists are below in the preference list of an agent. Note the asymmetry between agents corresponding to positive and negative literals: the positive (respectively, negative) literal agents have a dummy item as the least preferred item (respectively, the consistency items as the least preferred items) relevant to the picking in the choice round.

We say that agent 1 makes a *consistent choice* in the choice round corresponding to x_i if he picks both $o^1_{\neg x_i}$ and $o^2_{\neg x_i}$ (in which case we say he *assigns* x_i to true) or both $o^1_{x_i}$ and $o^2_{x_i}$ (in which case we say he *assigns* x_i to false). Given the preferences of other agents, the pairs that agent 1 can possibly get are only these three ones: $\{h^1_{x_i}, h^3_{\neg x_i}\}$, $\{h^1_{\neg x_i}, h^3_{x_i}\}$ and $\{h^2_{x_i}, h^2_{\neg x_i}\}$. If he makes a consistent choice then he will get one of the first two pairs, for which he has the same utility (see Tables 1 and 2), and if he makes an inconsistent choice then he will get the pair $\{h^2_{x_i}, h^2_{\neg x_i}\}$, which gives him a smaller utility than the other two pairs (see Tables 3 and 4).

Clause round The sequence in clause round corresponding to clause $c = (l_i \vee l_j \vee l_k)$ is as follows.

$$a_{l_i}, a_{l_j}, a_{l_k}, - - -, 1$$

For each literal l in clause c, there is an agent $a^1_{\neg l}$ or $a^2_{\neg l}$ that features in the round. Recall that each literal occurs in exactly two clauses in the set of clauses. Agent $a^1_{\neg l}$ (respectively, $a^2_{\neg l}$) features if c is the first (respectively, second) clause in which l is present.

After all the clause rounds are finished, agent 1 will get $|C|$ turns in which he will get the clause items that are still available.

For agent 1, according to u_1, his relevant preference in the clause round is to go for o^1_c.

For a literal l, let c and c' be the two clauses containing l. If $l = x$ (x being a propositional variable), the preferences of agents $a^1_{\neg x}$ and $a^2_{\neg x}$ in the round corresponding to l are

$$a^1_{\neg x}: \quad h^2_x, h^3_x, o^3_c, o^2_c, o^1_c$$
$$a^2_{\neg x}: \quad h^2_x, h^3_x, o^3_{c'}, o^2_{c'}, o^1_{c'}$$

and if $l = \neg x$ then the preferences of agents a^1_x and a^2_x in the round corresponding to l are

$$a^1_x: \quad d^{12}_{\neg x}, o^3_c, o^2_c, o^1_c$$
$$a^2_x: \quad d^{22}_{\neg x}, o^3_{c'}, o^2_{c'}, o^1_{c'}$$

Note that the consistency items h^2_x and h^3_x, as well as the dummy items $d^{12}_{\neg x}$ and $d^{22}_{\neg x}$, were possibly picked in the choice round corresponding to variable x.

If agent 1 has made a consistent choice in the choice round corresponding to variable x and has set x to "true", then $a^1_{x_i}$ and $a^2_{x_i}$ are now ready to get clause items in their respective clause rounds. If agent 1 has made a consistent choice in the choice round corresponding to variable x and has set x to "false", then $a^1_{\neg x_i}$ and $a^2_{\neg x_i}$ already want to get clause items in their respective clause rounds.

In the clause round, for any literal l that is not satisfied by the truth assignment resulting from the choice round, the agent corresponding to l gets a clause item o^i_c where c is a clause containing $\neg l$. For example, if x has been assigned to false, then $a^1_{\neg x}$ gets a clause item for the first clause in which

Stage	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Agent	1	$a^1_{\neg x_i}$	$a^2_{\neg x_i}$	$a^1_{x_i}$	$a^2_{x_i}$	1	$a^1_{\neg x_i}$	$a^2_{\neg x_i}$	$a^1_{x_i}$	$a^2_{x_i}$	$a^1_{\neg x_i}$	$a^2_{\neg x_i}$	1	$a^1_{x_i}$	$a^2_{x_i}$	1
Item picked	$o^1_{\neg x_i}$	$o^1_{x_i}$	$d^{21}_{x_i}$	$d^{11}_{\neg x_i}$	$d^{21}_{\neg x_i}$	$o^2_{\neg x_i}$	$d^{11}_{x_i}$	$o^2_{x_i}$	$h^1_{\neg x_i}$	$h^2_{\neg x_i}$	$d^{12}_{x_i}$	$d^{22}_{x_i}$	$h^3_{\neg x_i}$	$d^{12}_{\neg x_i}$	$d^{22}_{\neg x_i}$	$h^1_{x_i}$

Table 1: Choice round for variable x_i in which agent 1 makes consistent choice $\{o^1_{\neg x_i}, o^2_{\neg x_i}\}$ ("assign x_i to true"). Agents $a^1_{\neg x_i}$ and $a^2_{\neg x_i}$ next focus on items $h^2_{x_i}$ and $h^3_{x_i}$ before turning their attention to the clause items. On the other hand, $a^1_{x_i}$ and $a^2_{x_i}$ are now ready to get clause items. In this way, the agents corresponding to literals set to false are quicker to get their clause items.

Stage	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Agent	1	$a^1_{\neg x_i}$	$a^2_{\neg x_i}$	$a^1_{x_i}$	$a^2_{x_i}$	1	$a^1_{\neg x_i}$	$a^2_{\neg x_i}$	$a^1_{x_i}$	$a^2_{x_i}$	$a^1_{\neg x_i}$	$a^2_{\neg x_i}$	1	$a^1_{x_i}$	$a^2_{x_i}$	1
Item picked	$o^1_{x_i}$	$d^{11}_{x_i}$	$d^{21}_{x_i}$	$o^1_{\neg x_i}$	$d^{21}_{\neg x_i}$	$o^2_{x_i}$	$d^{12}_{x_i}$	$d^{22}_{x_i}$	$d^{11}_{\neg x_i}$	$o^2_{\neg x_i}$	$h^1_{x_i}$	$h^2_{x_i}$	$h^1_{\neg x_i}$	$h^2_{\neg x_i}$	$h^3_{\neg x_i}$	$h^3_{x_i}$

Table 2: Choice round for variable x_i in which agent 1 makes consistent choice $\{o^1_{x_i}, o^2_{x_i}\}$ ("assign x_i to false"). Agents $a^1_{x_i}$ and $a^2_{x_i}$ next focus on items $d^{12}_{\neg x_i}$ and $d^{22}_{\neg x_i}$ before turning their attention to the clause items. On the other hand, $a^1_{\neg x_i}$ and $a^2_{\neg x_i}$ are now ready to get clause items.

Stage	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Agent	1	$a^1_{\neg x_i}$	$a^2_{\neg x_i}$	$a^1_{x_i}$	$a^2_{x_i}$	1	$a^1_{\neg x_i}$	$a^2_{\neg x_i}$	$a^1_{x_i}$	$a^2_{x_i}$	$a^1_{\neg x_i}$	$a^2_{\neg x_i}$	1	$a^1_{x_i}$	$a^2_{x_i}$	1
Item picked	$o^1_{x_i}$	$d^{11}_{x_i}$	$d^{21}_{x_i}$	$o^1_{\neg x_i}$	$d^{21}_{\neg x_i}$	$o^2_{\neg x_i}$	$d^{12}_{x_i}$	$o^2_{x_1}$	$d^{11}_{\neg x_1}$	$h^1_{\neg x_i}$	$h^1_{x_i}$	$d^{22}_{x_i}$	$h^2_{\neg x_i}$	$h^2_{\neg x_i}$	$h^3_{\neg x_i}$	$h^2_{x_i}$

Table 3: Choice round for variable x_i in which agent 1 makes inconsistent choice $\{o^1_{x_i}, o^2_{\neg x_i}\}$. As a result of making an inconsistent choice, agent 1 does not get $h^1_{\neg x_i}$ nor $h^1_{x_i}$.

Stage	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Agent	1	$a^1_{\neg x_i}$	$a^2_{\neg x_i}$	$a^1_{x_i}$	$a^2_{x_i}$	1	$a^1_{\neg x_i}$	$a^2_{\neg x_i}$	$a^1_{x_i}$	$a^2_{x_i}$	$a^1_{\neg x_i}$	$a^2_{\neg x_i}$	1	$a^1_{x_i}$	$a^2_{x_i}$	1
Item picked	$o^1_{\neg x_i}$	$o^1_{x_i}$	$d^{21}_{x_i}$	$d^{11}_{\neg x_i}$	$d^{21}_{\neg x_i}$	$o^2_{x_i}$	$d^{11}_{x_i}$	$d^{22}_{x_i}$	$h^1_{\neg x_i}$	$o^2_{\neg x_i}$	$d^{12}_{x_i}$	$h^1_{x_i}$	$h^2_{\neg x_i}$	$d^{12}_{\neg x_i}$	$d^{22}_{\neg x_i}$	$h^2_{x_i}$

Table 4: Choice round for variable x_i in which agent 1 makes inconsistent choice $\{o^1_{\neg x_i}, o^2_{x_i}\}$.

x is present and a^2_x gets a clause item for the second clause in which x is present. Therefore, if all the literals of a clause c are false, then agent 1 does not get o^1_c.

If literal l is satisfied, the agent $a_{\neg l}$ corresponding to it gets a dummy or consistency item in that round instead of a clause item. This means that if all the literals of a clause are not satisfied, then all three clause items of a clause are gone, and agent 1 does not get a clause item. He is much more interested in the clause item o^1_c than in the clause items o^2_c and o^3_c. The other clause items are far down in his preference list so he would rather get all the top clause items o^1_c for each clause c rather than o^2_c and o^3_c.

For example, let us consider clause $c = (x_i \vee \neg x_j \vee \neg x_k)$, ans assume variables x_i, x_j and x_k are set to true, *i.e.*, agent 1 has picked choice items corresponding to $\neg x_i$, $\neg x_j$ and $\neg x_k$. This means that in the clause round, $a^1_{x_j}$ and $a^1_{x_k}$ are ready to take their clause items o^3_c and o^2_c but $a^1_{\neg x_i}$ wants to get one of the unallocated consistency items before he is interested in consistency items o^3_c, o^2_c, o^1_c. This is helpful for agent 1 because he can get o^1_c. Since each literal occurs exactly twice in the set of clauses, note that as long as 1 makes a consistent choice, there will be another clause c' in which literal x is present and if x is set to true, then $a^2_{\neg x}$ will get $h^3_{x_i}$ and hence 1 will be able to get $o_{c'}$. Note that after the clause rounds all the consistency items are already allocated, so agent 1 can hope to get all the top clause items if they were not already taken in the clause rounds.

Stage	1	2	3	after clause rounds
Agent	$a^1_{\neg x_i}$	$a^1_{x_j}$	$a^1_{x_k}$	1
Item picked	$h^2_{x_i}$	o^3_c	o^2_c	o^1_c

Table 5: Clause round corresponding to $c = (x_i \vee \neg x_j \vee \neg x_k)$

Collection round The sequence in this round is $\underbrace{1, \ldots, 1}_{|C|}$. At this stage, agent 1 prefers $o^1_{c_1}, \ldots, o^1_{c_{|C|}}$ to all other remaining items.

The idea is that if agent 1 make choices that sets all the clauses to true, then he gets all the clause items. Note that if 1 makes a consistent choice for the variables but does not pick up all the clause items in the collection round (because the set of clauses is unsatisfiable), then 1 does not get all the clause items. Since there are items less preferred by 1 than the o^1_cs such as o^2_c and o^3_cs, agent 1 is forced to pick a much less preferred item in the collection round.

Based on construction of the choice and clause rounds, we are in a position to prove a series of claims.

Claim 1. *If agent* 1 *does not make a consistent choice of the variable items, then he does not get utility* T.

Proof. If 1 does not make a choice at all in a choice round (that is, if he does not pick one of o^1_x or $o^1_{\neg x}$ and one of o^2_x or $o^2_{\neg x}$), then his most preferred items corresponding to the literals are then taken by the agents corresponding to the literal. If 1 makes a choice in each choice round but not a consistent one, then he gets $\{h^2_{\neg x}, h^2_x\}$ which has sufficiently less utility than $\{h^1_{\neg x}, h^3_x\}$ or $\{h^1_x, h^3_{\neg x}\}$ which implies he cannot get total utility T. □

Claim 2. *If agent* 1 *makes consistent choices but the assignment is not satisfying, then agent* 1 *does not get utility* T.

Proof. If some clause c is set to false, then agent 1 is not able to o^1_c because the literal agents in the clause round corresponding to c take all the items o_c^3, o_c^2, o_c^1. This implies that agent 1 does not get utility T. □

Claim 3. *If there exists a satisfying assignment for C, then agent 1 can get utility T.*

Proof. If there exists a satisfying assignment, then consider the preference report of agent 1 in which in each choice round, he picks $o_{x_i}^1$ and $o_{x_i}^2$ if x_i is set to be false. By doing this he gets to pick a top consistency item in that round as well. Since all the clauses are satisfied, in each clause round, agent 1 is able to get his clause item o_c^1. The utilities are set in a way so that as long as agent 1 gets two items corresponding to the same literal and hence at least one top choice consistency items in each round and his target clause item in each clause round, agent 1 gets utility at least T. □

The claims show that agent 1 gets utility at least T if and only if there is a satisfying truth assignment. □

The result above also gives us the following statement: computing a best response is NP-hard. This raises the following question: what is the complexity of testing whether there exists a report that gives more utility than the truthful report ?

Theorem 3. *Checking whether there exists a report that yields more utility than the truthful report is NP-complete.*

Proof. We first prove that the restriction of SAT to set of clauses such that (a) exactly one clause has only negative literals, and (b) each literal appears exactly twice in the set of clauses, is NP-complete. We show this by reduction from SAT with restriction (b), which we know to be NP-complete. Let $F = \{c_1, ..., c_n\}$ be the initial set of clauses, with (b) holding. Let G the following set of clauses, where $d_1, ..., d_n$ are new variables:

$$G = \{c_i \vee d_i, 1 \leq i \leq n\} \cup \{\neg d_i \vee d_{i+1}, 1 \leq i < n\} \cup \{\neg d_n\} \cup \{d_1 \vee \neg d_1 \vee ... \vee \neg d_n\}$$

Each literal (including d_i and $\neg d_i$ for all i) appears exactly twice in G, all clauses except one ($\neg d_n$) contain at least one positive literal. Now, G is equivalent to $\{\neg d_1, \ldots, \neg d_n\} \cup F$, therefore G is satisfiable if and only if F is satisfiable.

We now present a reduction from SAT with restrictions (a) and (b) to the existence of a better response than the truthful report. The reduction is essentially the same as that in the proof of Theorem 2, except that the clauses may not be of size 3. For clauses of different size, we simply create as many clause items as the number of literals in the clause and agent 1 still needs to get the top clause item of each clause to achieve his target utility. The reduction again involves choice rounds for all the variables and the goal is for the agent 1 to maximize utility by making consistent choices and then get as many clause items as possible.

In the same way as in the proof of Theorem 2, we can show that if the set of clauses is not satisfiable, agent 1's optimal strategy is to tell the truth (*i.e.*, set all variables to true) because this leads him to get consistent variable items and all clause items except for the last clause, and that if the set of clauses is satisfiable, then the satisfying truth assignment also gives the corresponding optimal preference report for agent 1. Therefore, checking whether there exists a better report for agent 1 than the truthful report is NP-complete. □

Manipulation under Binary Utilities

We now consider the restriction that the manipulator is asked to report a linear order but has *binary* preferences over items: the set of items is partitioned into two subsets O^+ and O^-, such that for each $o \in O^+$, $u_i(o) = \alpha$ and for each $o \in O^-$, $u_i(o) = \beta < \alpha$; that is, the items in O^+ are the most preferred items of agent 1. Binary utilities are important for *e.g.*, when each agent is only interested in a subset of acceptable items (Bogomolnaia and Moulin 2004). Since the number of items he gets with a fixed sequence is fixed, the utility obtained with a given report is maximal if and only if the number of objects in O^+ he gets is maximal. Therefore the manipulation problem under this restriction consists in finding a report leading agent 1 to get a maximum number of 'top items'.

Let us assume that agent 1 (the manipulator) has n_1 turns. Let π_{-1} the policy obtained by removing all occurrences of 1 in π, and let $First(\pi_{-1}, \succ_2, \ldots, \succ_n, O^+)$ be the first item in O^+ picked by some agent when simulating π_{-1} on $(\succ_2, \ldots, \succ_n)$. For example, take $\pi = 231312$, $\succ_2 = (o_1, o_2, o_3, o_4, o_5, o_6)$, $\succ_3 = (o_1, o_3, o_5, o_2, o_4, o_6)$ and $O^+ = \{o_2, o_4\}$. Then we have $\pi_{-1} = 2332$ and $First(\pi_{-1}, \succ_2, \succ_3, O^+) = o_2$, since simulating π_{-1} leads to 2 taking o_1, 3 taking o_3 and o_5, and then 2 taking o_2.

Moreover, let us write $\pi = (head(\pi), 1, tail(\pi))$, where $head(\pi)$ is the longest starting subsequence of π in which 1 does not occur, and $tail(\pi)$ is the subsequence of π starting right after the first occurrence of 1. Let $Allocate(head(\pi), \succ_2, \ldots, \succ_n)$ be the set of items allocated to $2, \ldots, n$ when simulating $head(\pi)$ with $(\succ_2, \ldots, \succ_n)$. For instance, with π, $\succ_2$ and $\succ_3$ as above, we have $head(\pi) = 23$, $tail(\pi) = 312$, and $Allocate(head(\pi), \succ_2, \succ_3) = \{o_1, o_3\}$.

Consider the following algorithm BR:

Input: O^+, π, $(\succ_2, \ldots, \succ_n)$

Output: $\succ'_1$

$k \leftarrow 1$;
$n_1 \leftarrow$ number of occurrences of 1 in π;
While $k \leq n_1$ and $O^+ \neq \emptyset$ and $\pi \neq 11\ldots1$
  $O^* \leftarrow Allocate(head(\pi), \succ_2, \ldots, \succ_n)$;
  remove O^* from O, O^+, and $(\succ_2, \ldots, \succ_n)$;
  $\pi \leftarrow tail(\pi)$;
  $a_k \leftarrow First(\pi_{-1}, \succ_2, \ldots, \succ_n, O^+)$;
  remove a_k from O, O^+, and $(\succ_2, \ldots, \succ_n)$;
  $k \leftarrow k + 1$;
End While
$\succ'_1 \leftarrow (a_1, \ldots, a_k)$;
complete $\succ'_1$ with the remaining items of O^+ (if any) in an arbitrary way, and then by other items in an arbitrary way;
Return $\succ'_1$

Theorem 4. *A best response can be computed in polynomial time when the manipulator has binary utilities: Algorithm BR outputs a best response.*

Proof. Assume that agent 1 has n_1 turns. 1 is interested in getting a maximum number n_1' top items from O^+. Let $\succ_1'$ be the sequence of objects obtained using Algorithm BR.

The main idea of the algorithm is that $\succ_1'$ is gradually built in a way so that 1 can get all the top items in the list. A new top item a_k is only appended to the list if 1 can additionally get it along with the previous items in the list. Just before adding a_k, when we simulate sequential allocation, 1 does not get a_k because someone else gets it right after 1's turn in which 1 abstained from picking an item because $\succ_1'$ only had $k-1$ items. Hence if 1 does not abstain and in fact makes use of his k-th turn, then 1 can get a_k.

To prove the optimality of the algorithm, we reason iteratively on the number of picking turns of agent 1. Namely, we prove the following statement: (H_{n_1}) suppose that the number of remaining picking turns of 1 is n_1. Suppose that there is a picking strategy for agent 1 to get c objects among those from O^+, and let $\succ_1''= \langle b_1, \ldots, b_{n_1}\rangle$ be the sequence of objects he gets with this strategy. Then the strategy $\succ_1'''= \langle a_1, b_2, \ldots, b_{n_1}\rangle$, where we have simply replaced b_1 by a_1 in $\succ_1''$ gives at least c objects from O^+ to agent 1.

We omit the argument of (H_{n_1}) due to lack of space.

By successively applying this hypothesis on $\langle b_{n_1}\rangle$, $\langle b_{n_1-1}, b_{n_1}\rangle$,... we prove that for each k, the strategy $\langle a_k, \ldots, a_{n_1}\rangle$ gives to agent 1 as many objects from O^+ as the strategy $\langle b_k, \ldots, b_{n_1}\rangle$. This proves that for each arbitrary picking strategy, the strategy given by the algorithm gives to agent 1 at least as many objects from O^+. Hence, the algorithm returns an optimal strategy. □

Manipulation under responsive preferences

An allocation S is *more preferred with respect to responsive (RS) preferences* than allocation T if S is a result of replacing an item in T with a strictly more preferred item (Aziz et al. 2015b; Brams and King 2005, for instance). Note that the responsive relation is transitive but not complete.

We examine the problem of determining whether there exists an untruthful report which yields an allocation that is strictly more preferred with respect to responsive preferences. The problem is equivalent to checking whether there exists a report that yields more utility with respect to *all* additive utilities consistent with the ordinal preferences. In contrast to the problem for specific utilities, this particular problem of checking whether there exists such a clear manipulation can be solved in polynomial time. Our algorithm is based on an intimate connection that we identify between manipulation under responsive preferences and under lexicographic preferences.

Theorem 5. *It can be checked in polynomial time whether there exists a manipulation that gives an allocation that is strictly more preferred with respect to responsive set extension than the truthful outcome.*

Proof. We first present the algorithm.

Input: picking sequence π, profile $(\succ_1, \succ_2, \ldots, \succ_n)$
Output: $\succ_1'$

Rename items such that $\succ_1 = o_1, \ldots, o_m$
$O_1 \leftarrow$ set of items obtained by agent 1 with his sincere strategy;
$k \leftarrow 1$; $O^* \leftarrow \emptyset$;
Repeat
  If agent 1 can obtain $O^* \cup \{o_k\}$ **then**
    $O^* \leftarrow O^* \cup \{o_k\}$
  endif
  $k \leftarrow k+1$
Until O^* lexicographically dominates O_1 **or** $k > m$
If O^* lexicographically dominates O_1 **then**
  $\succ_1' \leftarrow$ ranking of O^* allowing agent 1 to obtain O^*
  Complete $\succ_1'$ with the items in $O \setminus O^*$, ranked as in $\succ_1$
  Return $\succ_1'$
else Return 'failure'
endif

It can be determined in polynomial time whether agent 1 can obtain $O^* \cup \{o_k\}$ (Bouveret and Lang 2011, Proposition 7), therefore the algorithm works in polynomial time. Now we claim that when the algorithm returns $\succ_1'$, then $\succ_1'$ yields a strictly better allocation than the truthful outcome. If the truthful report yields the lexicographical optimal outcome, then clearly, there can be no responsively better outcome because a responsively better outcome is also a lexicographically better outcome. Now assume that the truthful outcome is lexicographically not the best. Then, there exists another allocation that is achievable that is lexicographically better. Consider the first (most preferred) item that is in the lexicographically more preferred outcome but not in the truthful outcome. We compute an allocation and partial preference of the manipulator that does not include any less preferred items. For the remaining items, we simply append them in the manipulator's preference list in decreasing order of preference. We claim that the outcome responsively dominates the truthful outcome from the manipulating agent's truthful preferences. The argument is technical and long and is omitted due to lack of space. □

Example 6. *Let $n = 3$, $m = 7$, $\pi = 1231231$, and*

$\succ_1$: $o_1, o_2, o_3, o_4, o_5, o_6, o_7$
$\succ_2$: $o_2, o_1, o_5, o_3, o_4, o_6, o_7$
$\succ_3$: $o_2, o_3, o_4, o_1, o_5, o_7, o_6$

We have $O_1 = \{o_1, o_4, o_6\}$. At the first step, o_1 is added to O^*; at the next step, o_2 is not added since $\{o_1, o_2\}$ is not feasible; at the next step, o_3 is added; as $O^* = \{o_1, o_3\}$ lexicographically dominates O_1, we exit the loop, initialize $\succ_1'$ to (o_3, o_1), and complete it into $\succ_1'$: $o_3, o_1, o_2, o_4, o_5, o_6, o_7$ which leads agent 1 to obtain $\{o_1, o_3, o_6\}$.

Conclusion

In this paper, we showed that computing a best response under sequential allocation to maximize additive utility is NP-hard. We also gave a contrasting results that manipulating sequential allocation under binary utilities and also with respect to responsive preferences is easy. Our NP-hardness result does not involve a constant number of agents. It remains an interesting open problem whether manipulating sequential allocation with respect to cardinal utilities is NP-hard when the number of agents is three or some other constant.

Acknowledgement This work is partly supported by the project ANR-14-CE24-0007-01 “CoCoRICo-CoDec”.

References

Aziz, H.; Gaspers, S.; Mackenzie, S.; Mattei, N.; Narodytska, N.; and Walsh, T. 2015a. Manipulating the probabilistic serial rule. In *Proc. of 14th AAMAS Conference*, 1451–1459.

Aziz, H.; Gaspers, S.; Mackenzie, S.; and Walsh, T. 2015b. Fair assignment of indivisible objects under ordinal preferences. *Artificial Intelligence* 227:71–92.

Aziz, H.; Walsh, T.; and Xia, L. 2015. Possible and necessary allocations via sequential mechanisms. In *Proc. of 24th IJCAI*, 468–474.

Berman, P.; Karpinski, M.; and Scott, A. D. 2003. Approximation hardness of short symmetric instances of MAX-3SAT. *Electronic Colloquium on Computational Complexity (ECCC)* (049).

Bogomolnaia, A., and Moulin, H. 2004. Random matching under dichotomous preferences. *Econometrica* 72(1):257–279.

Bouveret, S., and Lang, J. 2011. A general elicitation-free protocol for allocating indivisible goods. In *Proc. of 22nd IJCAI*, 73–78. AAAI Press.

Bouveret, S., and Lang, J. 2014. Manipulating picking sequences. In *Proc. of 21st ECAI*, 141–146.

Brams, S. J., and King, D. L. 2005. Efficient fair division: Help the worst off or avoid envy? *Rationality and Society* 17(4):387–421.

Brams, S. J., and Straffin, P. D. 1979. Prisoners’ dilemma and professional sports drafts. *The American Mathematical Monthly* 86(2):80–88.

Brams, S. J., and Taylor, A. D. 1996. *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge University Press.

Kalinowski, T.; Narodytska, N.; Walsh, T.; and Xia, L. 2013. Strategic behavior when allocating indivisible goods sequentially. In *Proc. of 27th AAAI Conference*, 452–458. AAAI Press.

Kalinowski, T.; Narodytska, N.; and Walsh, T. 2013. A social welfare optimal sequential allocation procedure. In *Proc. of 22nd IJCAI*, 227–233. AAAI Press.

Kohler, D. A., and Chandrasekaran, R. 1971. A class of sequential games. *Operations Research* 19(2):270–277.

Levine, L., and Stange, K. E. 2012. How to make the most of a shared meal: Plan the last bite first. *The American Mathematical Monthly* 119(7):550–565.

Tominaga, Y.; Todo, T.; and Yokoo, M. 2016. Manipulations in two-agent sequential allocation with random sequences. In *Proc. of 15th AAMAS Conference*. IFAAMAS. 141–149.

Complexity of Manipulation

Example of the Reduction

Example 7. *We illustrate the reduction in the proof of Theorem 2. For the following SAT formula, we illustrate how we build an allocation setting with the agent set, item set, preferences of agents and the picking sequence.*

$$\underbrace{(x_1 \vee x_2 \vee x_3)}_{c_1} \underbrace{(\neg x_1 \vee \neg x_2 \vee \neg x_3)}_{c_2} \underbrace{(x_1 \vee \neg x_2 \vee x_3)}_{c_3} \underbrace{(\neg x_1 \vee x_2 \vee \neg x_3)}_{c_4}$$

Set of agents is $N = \{1\} \cup \{a^1_{x_i}, a^2_{x_i}, a^1_{\neg x_i}, a^2_{\neg x_i} : i \in \{1,2,3\}\}$. *Set of items is* $O = \{o^1_{x_i}, o^2_{x_i}, o^1_{\neg x_i}, o^2_{\neg x_i}, h^1_{x_i}, h^2_{x_i}, h^1_{\neg x_i}, h^2_{\neg x_i}, d^{11}_{x_i}, d^{12}_{x_i}, d^{21}_{x_i}, d^{22}_{x_i}, d^{11}_{\neg x_i}, d^{12}_{\neg x_i}, d^{21}_{\neg x_i}, d^{22}_{\neg x_i} : i \in \{1,2,3\}\}$

$$\begin{array}{rl}
1: & o^1_{x_1}, o^1_{\neg x_1}, o^2_{x_1}, o^2_{\neg x_1}, h^1_{\neg x_1}, h^2_{\neg x_1}, h^3_{\neg x_1}, h^1_{x_1}, h^2_{x_1}, h^3_{x_1}, \\
 & o^1_{x_2}, o^1_{\neg x_2}, o^2_{x_2}, o^2_{\neg x_2}, h^1_{\neg x_2}, h^2_{\neg x_2}, h^3_{\neg x_2}, h^1_{x_2}, h^2_{x_2}, h^3_{x_2} \\
 & o^1_{x_3}, o^1_{\neg x_3}, o^2_{x_3}, o^2_{\neg x_3}, h^1_{\neg x_3}, h^2_{\neg x_3}, h^3_{\neg x_3}, h^1_{x_3}, h^2_{x_3}, h^3_{x_3} \\
 & o^1_{c_1}, o^1_{c_2}, o^1_{c_3}, o^1_{c_4} \\
\hline
a^1_{\neg x_1}: & o^1_{x_1}, d^{11}_{x_1}, d^{12}_{x_1}, o^2_{x_1}, h^1_{x_1}, h^2_{x_1}, h^3_{x_1}, o^3_{c_1}, o^2_{c_1}, o^1_{c_1} \\
a^2_{\neg x_1}: & d^{21}_{x_1}, o^1_{x_1}, o^2_{x_1}, d^{22}_{x_i}, h^1_{x_1}, h^2_{x_1}, h^3_{x_1}, o^3_{c_3}, o^2_{c_3}, o^1_{c_3} \\
a^1_{\neg x_2}: & o^1_{x_2}, d^{11}_{x_2}, d^{12}_{x_2}, o^2_{x_2}, h^1_{x_2}, h^2_{x_2}, h^3_{x_2}, o^3_{c_1}, o^2_{c_1}, o^1_{c_1} \\
a^2_{\neg x_2}: & d^{21}_{x_2}, o^1_{x_2}, o^2_{x_2}, d^{22}_{x_i}, h^1_{x_2}, h^2_{x_2}, h^3_{x_2}, o^3_{c_4}, o^2_{c_4}, o^1_{c_4} \\
a^1_{\neg x_3}: & o^1_{x_3}, d^{11}_{x_3}, d^{12}_{x_3}, o^2_{x_3}, h^1_{x_3}, h^2_{x_3}, h^3_{x_3}, o^3_{c_1}, o^2_{c_1}, o^1_{c_1} \\
a^2_{\neg x_3}: & d^{21}_{x_3}, o^1_{x_3}, o^2_{x_3}, d^{22}_{x_i}, h^1_{x_3}, h^2_{x_3}, h^3_{x_3}, o^3_{c_3}, o^2_{c_3}, o^1_{c_3} \\
\hline
a^1_{x_1}: & o^1_{\neg x_1}, d^{11}_{\neg x_1}, h^1_{\neg x_1}, o^2_{\neg x_1}, h^2_{\neg x_1}, h^3_{\neg x_1}, d^{12}_{\neg x_1} o^3_{c_2}, o^2_{c_2}, o^1_{c_2} \\
a^2_{x_1}: & d^{21}_{\neg x_1}, o^1_{\neg x_1}, o^2_{\neg x_1}, h^1_{\neg x_1}, h^2_{\neg x_1}, h^3_{\neg x_1}, d^{22}_{\neg x_1}, o^3_{c_4}, o^2_{c_4}, o^1_{c_4} \\
a^1_{x_2}: & o^1_{\neg x_2}, d^{11}_{\neg x_2}, h^1_{\neg x_2}, o^2_{\neg x_2}, h^2_{\neg x_2}, h^3_{\neg x_2}, d^{12}_{\neg x_2}, o^3_{c_2}, o^2_{c_2}, o^1_{c_2} \\
a^2_{x_2}: & d^{21}_{\neg x_2}, o^1_{\neg x_2}, o^2_{\neg x_2}, h^1_{\neg x_2}, h^2_{\neg x_2}, h^3_{\neg x_2}, d^{22}_{\neg x_2}, o^3_{c_3}, o^2_{c_3}, o^1_{c_3} \\
a^1_{x_3}: & o^1_{\neg x_3}, d^{11}_{\neg x_3}, h^1_{\neg x_3}, o^2_{\neg x_3}, h^2_{\neg x_3}, h^3_{\neg x_3}, d^{12}_{\neg x_3}, o^3_{c_2}, o^2_{c_2}, o^1_{c_2} \\
a^2_{x_3}: & d^{21}_{\neg x_3}, o^1_{\neg x_3}, o^2_{\neg x_3}, h^1_{\neg x_3}, h^2_{\neg x_3}, h^3_{\neg x_3}, d^{22}_{\neg x_3}, o^3_{c_4}, o^2_{c_4}, o^1_{c_4} \\
\hline
\end{array}$$

The picking sequence is as follows

$$\begin{array}{rl}
\textit{choice round 1} & 1, a^1_{\neg x_1}, a^2_{\neg x_1}, a^1_{x_1}, a^2_{x_1}, 1, a^1_{\neg x_1}, a^2_{\neg x_1}, \\
 & a^1_{\neg x_1}, a^2_{\neg x_1}, a^1_{x_1}, a^2_{x_1}, 1, a^1_{x_1}, a^2_{x_1}, 1 \\
\textit{choice round 2} & 1, a^1_{\neg x_2}, a^2_{\neg x_2}, a^1_{x_2}, a^2_{x_2}, 1, a^1_{\neg x_2}, \\
 & a^2_{\neg x_2}, a^1_{\neg x_2}, a^2_{\neg x_2}, a^1_{x_2}, a^2_{x_2}, 1, a^1_{x_2}, a^2_{x_2}, 1 \\
\textit{choice round 3} & 1, a^1_{\neg x_3}, a^2_{\neg x_3}, a^1_{x_3}, a^2_{x_3}, 1, a^1_{\neg x_3}, a^2_{\neg x_3}, \\
 & a^1_{\neg x_3}, a^2_{\neg x_3}, a^1_{x_3}, a^2_{x_3}, 1, a^1_{x_3}, a^2_{x_3}, 1 \\
\textit{clause round 1} & a^1_{\neg x_1}, a^1_{\neg x_2}, a^1_{\neg x_3} \\
\textit{choice round 2} & a^1_{x_1}, a^1_{x_2}, a^1_{x_3} \\
\textit{choice round 3} & a^2_{\neg x_1}, a^2_{x_2}, a^2_{\neg x_3} \\
\textit{choice round 4} & a^2_{x_1}, a^2_{\neg x_2}, a^2_{x_3} \\
\textit{collection round} & 1, 1, 1, 1
\end{array}$$

The formula is satisfiable if x_1 is true, x_2 is false and x_3 is false. Let us show how the allocation looks like when agent 1 picks items according to the truth assignment.

Manipulation under Binary Utilities

Complete Proof of Theorem 4

Proof. Let us assume that the manipulator is agent 1 and has n_1 turns. The agent is interested in getting a maximum number n'_1 top items from set $O^+ = \max_{\succ_1}(O)$.

Let $\succ'_1$ be the sequence of objects obtained using the following algorithm.

> We set index k to 1 and agent 1's preference list $\succ'_1$ to empty and gradually update the list. For $k = 1$ to n_i, we do the following. We simulate sequential allocation with profile $(\succ'_1, \succ_{N\setminus\{1\}})$ (Since $\succ'_1$ is incomplete, if an agent 1's turn comes and all items in his list have already been picked, then the agent does not pick any item). If there is no item in O^+ that is picked by some other agent after agent 1's k-th turn, then simply append all the unallocated top items to $\succ'_1$ in an arbitrary order followed by all other non-top items and return $\succ'_1$. Otherwise, let a_k be the first item from O^+ that is taken by some other agent after agent 1's k turns. We append a_k to $\succ'_1$. Increment k.

The main idea of the algorithm is that $\succ'_1$ is gradually built in a way so that 1 can get all the top items in the list. A new top item a_k is only appended to the list if 1 can additionally get it along with the previous items in the list. Just before adding a_k, when we simulate sequential allocation, 1 does not get a_k because someone else gets it right after 1's turn in which 1 abstained from picking an item because $\succ'_1$ only had $k-1$ items. Hence if 1 does not abstain and in facts makes use of his k-th turn, then 1 can get a_k.

To prove the optimality of the algorithm, we will reason iteratively on the number of picking turns of agent 1. Namely, we can prove the following statement:

> (H_{n_1}) suppose that the number of remaining picking turns of agent 1 is n_1. Suppose that there is a picking strategy for agent 1 to get c objects among those from O^+, and let $\succ''_1 = \langle b_1, \ldots, b_{n_1} \rangle$ be the sequence of objects she gets with this strategy. Then the strategy $\succ'''_1 = \langle a_1, b_2, \ldots, b_{n_1} \rangle$, where we have simply replaced b_1 by a_1 in $\succ''_1$ gives at least c objects from O^+ to agent 1.

1. Suppose that $a_1 \notin \{b_2, \ldots, b_{n_1}\}$. It means that at each step, each agent $k \neq 1$ either picks the same object as before, or an object one rank higher (it is so if some agent has picked b_1, leaving another object to pick). Hence, objects $\{b_2, \ldots, b_{n_1}\}$ are still available for agent 1. If $b_1 \in O^+$, then $|O^+ \cap \{b_2, \ldots, b_{n_1}\}| = c - 1$, and hence agent 1 still gets c objects from O^+ with strategy $\langle a_1, \ldots, b_{n_1} \rangle$. Otherwise, $|O^+ \cap \{b_2, \ldots, b_{n_1}\}| = c$, and hence agent 1 gets $c + 1$ objects with strategy $\langle a_1, \ldots, b_{n_1} \rangle$.
2. Suppose that $\exists k > 1$ such that $a_1 = b_k$. By a similar argument as above, agent 1 can still pick all the objects from b_2 to b_{k-1}. Now two cases can happen:

Stage	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Agent	1	$a^1_{\neg x_1}$	$a^2_{\neg x_1}$	$a^1_{x_1}$	$a^2_{x_1}$	1	$a^1_{\neg x_1}$	$a^2_{\neg x_1}$	$a^1_{x_1}$	$a^2_{x_1}$	$a^1_{\neg x_1}$	$a^2_{\neg x_1}$	1	$a^1_{x_1}$	$a^2_{x_1}$	1
Item picked	$o^1_{\neg x_1}$	$o^1_{x_1}$	$d^{21}_{x_1}$	$d^{11}_{\neg x_1}$	$d^{21}_{\neg x_1}$	$o^2_{\neg x_1}$	$d^{11}_{x_1}$	$o^2_{x_1}$	$h^1_{\neg x_1}$	$h^2_{\neg x_1}$	$d^{12}_{x_1}$	$d^{22}_{x_1}$	$h^3_{\neg x_1}$	$d^{12}_{\neg x_1}$	$d^{22}_{\neg x_1}$	$h^1_{x_1}$

Table 6: Choice round 1 for variable x_1 in which agent 1 makes consistent choice $o^1_{\neg x_1}$ and $o^2_{\neg x_1}$.

Stage	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Agent	1	$a^1_{\neg x_2}$	$a^2_{\neg x_2}$	$a^1_{x_2}$	$a^2_{x_2}$	1	$a^1_{\neg x_2}$	$a^2_{\neg x_2}$	$a^1_{x_2}$	$a^2_{x_2}$	$a^1_{\neg x_2}$	$a^2_{\neg x_2}$	1	$a^1_{x_2}$	$a^2_{x_2}$	1
Item picked	$o^1_{x_2}$	$d^{11}_{x_2}$	$d^{21}_{x_2}$	$o^1_{\neg x_2}$	$d^{21}_{\neg x_2}$	$o^2_{x_2}$	$d^{12}_{x_2}$	$d^{22}_{x_2}$	$d^{11}_{\neg x_2}$	$o^2_{\neg x_2}$	$h^1_{x_2}$	$h^2_{x_2}$	$h^1_{\neg x_2}$	$h^2_{\neg x_2}$	$h^3_{\neg x_2}$	$h^3_{x_2}$

Table 7: Choice round 2 for variable x_2 in which agent 1 makes consistent choice $o^1_{x_2}$ and $o^2_{x_2}$ so that variable x_2 is set to false.

Stage	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Agent	1	$a^1_{\neg x_3}$	$a^2_{\neg x_3}$	$a^1_{x_3}$	$a^2_{x_3}$	1	$a^1_{\neg x_3}$	$a^2_{\neg x_3}$	$a^1_{x_3}$	$a^2_{x_3}$	$a^1_{\neg x_3}$	$a^2_{\neg x_3}$	1	$a^1_{x_3}$	$a^2_{x_3}$	1
Item picked	$o^1_{x_3}$	$d^{11}_{x_3}$	$d^{21}_{x_3}$	$o^1_{\neg x_3}$	$d^{21}_{\neg x_3}$	$o^2_{x_3}$	$d^{12}_{x_3}$	$d^{22}_{x_3}$	$d^{11}_{\neg x_3}$	$o^2_{\neg x_3}$	$h^1_{x_3}$	$h^2_{x_3}$	$h^1_{\neg x_3}$	$h^2_{\neg x_3}$	$h^3_{\neg x_3}$	$h^3_{x_3}$

Table 8: Choice round 3 for variable x_3 in which agent 1 makes consistent choice $o^1_{x_3}$ and $o^2_{x_3}$ so that variable x_3 is set to false.

Stage	1	2	3
Agent	$a^1_{\neg x_1}$	$a^1_{\neg x_2}$	$a^1_{\neg x_3}$
Item picked	$h^2_{x_1}$	$o^3_{c_1}$	$o^3_{c_1}$

Table 9: Clause round 1

Stage	1	2	3
Agent	$a^1_{x_1}$	$a^1_{x_2}$	$a^1_{x_3}$
Item picked	$o^3_{c_2}$	$d^{12}_{\neg x_2}$	$d^{12}_{\neg x_3}$

Table 10: Clause round 2

Stage	1	2	3
Agent	$a^2_{\neg x_1}$	$a^2_{x_2}$	$a^2_{\neg x_3}$
Item picked	$h^3_{x_1}$	$d^{22}_{\neg x_2}$	$o^2_{c_3}$

Table 11: Clause round 3

Stage	1	2	3
Agent	$a^2_{x_1}$	$a^2_{\neg x_2}$	$a^2_{x_3}$
Item picked	$o^2_{c_4}$	$o^2_{c_4}$	$d^{22}_{\neg x_3}$

Table 12: Clause round 4

Stage	1	2	3	4
Agent	1	1	1	1
Item picked	$o^1_{c_1}$	$o^1_{c_2}$	$o^1_{c_3}$	$o^1_{c_4}$

Table 13: Collection round

(a) b_1 is still available at agent 1's k^{th} turn. It then means that after this turn, the set of remaining available objects is exactly the same with $\succ'_1$ and $\succ''_1$. Hence, agent 1 picks exactly the same objects with both strategies, and hence still gets c objects from O^+.

(b) b_1 is not available anymore at agent 1's k^{th} turn, and suppose that b_1 has been picked by agent i at turn l. It means that during the execution of sequential allocation with $\succ'_1$ as agent 1's picking strategy, object $a_1 = b_k$ would have been picked after turn l (otherwise it would not have been available for agent 1). If $b_1 \in O^+$, this would contradict the definition of a_1, being the first object from O^+ which would have been picked by another agent at the beginning of the sequence. Hence $b_1 \notin O^+$. We can remark that agent 1 can still get all the objects from b_{k+1} to b_{n_1} (in turns k to $n_1 - 1$), and get one additional object at turn n_1. Altogether, she will thus get all the objects from $\{b_2, \dots, b_{n_1}\}$. Since $b_1 \notin O^+$, this set already contains c objects from O^+.

Hence we have proved (H_{n_1}). By successively applying this hypothesis on $\langle b_{n_1} \rangle$, $\langle b_{n_1-1}, b_{n_1} \rangle$,..., we prove that for each k, the strategy $\langle a_k, \dots, a_{n_1} \rangle$ gives to agent 1 as many objects from O^+ as the strategy $\langle b_k, \dots, b_{n_1} \rangle$. This proves that for each arbitrary picking strategy, the strategy given by the algorithm gives to agent 1 at least as many objects from O^+. Hence, the algorithm returns an optimal strategy. $\square$

Example 8. *Let $n = 3$, $m = 9$, $\pi = 123123123$, $\succ_2 = (o_4, o_5, o_6, o_1, o_2, o_3, o_7, o_8, o_9)$, $\succ_3 = (o_4, o_6, o_2, o_3, o_5, o_1, o_7, o_8, o_9)$, and $O^+ = \{o_1, o_2, o_3\}$.*

At iteration 1, $a_1 = First(\pi_{-1}, \succ_2, \succ_3, O^+) = 2$: 2 picks o_4, 3 picks o_6, 2 picks o_5, and 3 picks o_2. We have now $a_1 = o_2$, $O^+ = \{o_1, o_3\}$ and $\pi = 23123123$. Then $O^* = \{o_4, o_6\}$, $\succ_2 = (o_5, o_1, o_3, o_7, o_8, o_9)$, $\succ_3 = (o_3, o_5, o_1, o_7, o_8, o_9)$ and $\pi = 123123$.

At iteration 2, $a_2 = o_3$, $\succ_1 = (o_2, o_3)$, $O^+ = \{o_1\}$; then $head(\pi) = 23$, then $\pi = 123123$, At iteration 3, $head(\pi) = 23$, $O^* = \{o_1, o_5\}$, $\succ_2 = (o_7, o_8, o_9)$, $\succ_3 = (o_7, o_8, o_9)$, and then $O^+ = \emptyset$ and $\pi = 123$. Since $O^+ = \emptyset$, we stop and complete $\succ_1$ into, for instance, $(o_2, o_3, o_1, o_4, o_5, o_6, o_7, o_8, o_9)$.

Manipulation under responsive preferences

Complete Proof of Theorem 5

Proof. We first summarize the algorithm.

> Compute the lexicographic optimal allocation for agent 1 by the algorithm of Bouveret and Lang (Bouveret and Lang 2011). If the allocation is the same as the truthful outcome, there exists no manipulation that is better than the truthful allocation with respect to responsive preferences so return "No". Otherwise, iteratively compute the lexicographic best achievable allocation of increasing sizes as is done in the lexicographic manipulation algorithm of Bouveret and Lang (Bouveret and Lang 2011). Stop when there exists an achievable allocation that is lexicographically better than the truthful allocation. Such an allocation and its corresponding preference list may be partial. In such a case, simply append the remaining items in decreasing order of preferences to the optimal report preference and return this preference $\succ'$.

We claim that $\succ'$ yields an allocation that is strictly more preferred than the truthful outcome. If the truthful report yields the lexicographical optimal outcome, then clearly, there can be no responsively better outcome because a responsively better outcome is also a lexicographically better outcome. Now assume that truthful outcome is not lexicographically the best. Then, there exists another allocation that is achievable that is lexicographically better. Consider the first (most preferred) item that is in the lexicographically more preferred outcome but not in the truthful outcome. We compute an allocation and partial preference of the manipulator that does not include any less preferred items. For the remaining items, we simply append them in the manipulator's preference list in decreasing order of preference. We claim that the outcome responsively dominates the truthful outcome from the manipulating agent's truthful preferences. The argument is technical and long and we omit due to lack of space. Assume that in the truthful profile, agent 1 gets allocation $\{a_1, \ldots, a_k\}$ in the order $a_1, \ldots, a_k$. Now let us assume that the allocation is lexicographically not the best possible outcome for agent 1. Then there exists another allocation that lexicographically dominates $\{a_1, \ldots, a_k\}$. Let a_i be the most preferred item that can be replaced by strictly more preferred item b and let $X = O \setminus \{a_1, \ldots, a_k, b\}$ In that case, we know that the allocation $\{a_1, \ldots, a_{c-1}, b\}$ is achievable in the first c picks of agent 1 by a suitable untruthful report.

$$\underbrace{\{a_1, \ldots, a_{c-1}, b\}}_{\text{1st } i \text{ picks of agent 1}}$$

We first consider a modification $\succ^*$ of $\succ_1$ that yields the same allocation for agent 1. In $\succ_1^*$, 1 reports the items in $\{a_1, \ldots, a_{c-1}\}$ in $SA(\succ_1', \succ_{-1})$ in the same order as in $\succ_1'$. Since the items in $\{a_1, \ldots, a_{c-1}\}$ are achievable via report $\succ_1'$, they are achievable via report $\succ_1^*$ because each of the corresponding item is picked either at the same time or one pick earlier. Hence agent 1's allocation after reporting $\succ_1^*$ and $\succ_1$ is the same. Let us assume that agent 1 picks items in $\{a_1, \ldots, a_{c-1}\}$ in order $b_1, \ldots, b_{c-1}$.

Stage	1	2	t_1 =3	4	5	6	7	8	9	10
Item picked in $\succ'$	b_1	b_2	b	b_3	b_4	a_5	a_6	a_7	a_8	a_{10}
Item picked in $\succ^*$	b_1	b_2	b_3	b_4	a_5	a_6	a_7	a_8	a_9	a_{10}

Table 14: Illustrative allocation sequence for $c = 5$: agent 1 manages to get b instead of a_5.

We keep track of R' and R the set of unallocated items when 1 reports $\succ_1'$ and $\succ_1^*$ respectively. We consider the following phases:

$$----t_1----t_2----t_3---$$

We specify what t_1, t_2 and t_3 are. Let t_1 be the time step in which agent 1 picks b in $SA(\succ_1', \succ_{-1})$. Let t_2 be the time step in which some agent other than 1 picks b in $SA(\succ_1^*, \succ_{-1})$. Let t_3 be time step in which some agent other than 1 picks an item from $\{a_c, \ldots, a_k\}$. Note that since 1 has a total of k picks, it cannot pick all items from $\{a_c, \ldots, a_k\}$ given that it already picks $\{b, a_1, \ldots, a_{c-1}\}$. Now consider the turn number in which agent 1 picks b in $SA(\succ_1', \succ_{-1})$. Before this turn, there is no difference in $SA(\succ_1', \succ_{-1})$ and $SA(\succ_1^*, \succ_{-1})$. We now iteratively compare R and R'.

In step t_1, agent 1 picks a different item b_j in $SA(\succ_1^*, \succ_{-1})$. So $R' = R \cup \{b_j\} \setminus \{b\}$. Therefore in each subsequent step i between t_1 and t_2, the following happens:

1. if the turn is by some agent j other than 1, he picks the same items in both $SA(\succ_1', \succ_{-1})$ and $SA(\succ_1^*, \succ_{-1})$. In this case, $R' = R \cup \{b_j\} \setminus \{b\}$ so the difference between R' and R does not change.
2. if the turn is by 1, he picks $b_{j'-1}$ in $SA(\succ_1', \succ_{-1})$ but picks $b_{j'}$ in $SA(\succ_1^*, \succ_{-1})$. In that case, $R' = R \cup \{b_{j'}\} \setminus \{b\}$. So the difference between R' and R changes but $R' = R \cup \{b_{j'-1}\} \setminus \{b\}$ is changed to $R' = R \cup \{b_{j'}\} \setminus \{b\}$.

The pattern of the difference between R' and R remains like this until we reach turn t_2 when some agent j picks up b in $SA(\succ_1^*, \succ_{-1})$. The same agent j cannot pick b in $SA(\succ_1', \succ_{-1})$ because it was already picked by 1 t_1. Therefore, j picks some other item say x_1 in $SA(\succ_1', \succ_{-1})$. In that case $R' = R \cup \{a_{j'}\} \setminus \{x_1\}$. Note that x_1 cannot be any of the items in $\{a_1, \ldots, a_{c-1}\}$. It could possibly be a_c. Now let us consider the next turn. If it is another agent j who most prefers item $y \notin \{a_j', x_1\}$ in $SA(\succ_1^*, \succ_{-1})$ will also pick y

in $SA(\succ_1', \succ_{-1})$. It could be that agent j wants to pick x_1 in $SA(\succ_1^*, \succ_{-1})$ as it is extra item not available in $SA(\succ_1^*, \succ_{-1})$. In $SA(\succ_1', \succ_{-1})$, the same agent j may want to pick some other item x_2 because x_1 is already allocated and x_2 is the next most preferred unallocated item. In this case after the turn $R' = R \cup \{a_{j'}\} \setminus \{x_2\}$. If it is agent 1's turn he picks $a_{j'}$ in $SA(\succ_1', \succ_{-1})$ but picks $a_{j'+1}$ in $SA(\succ_1^*, \succ_{-1})$. In that case, $R' = R \cup \{a_{j'}\} \setminus \{x\}$. So the difference between R' and R changes but simply $R' = R \cup \{a_{j'-1}\} \setminus \{x\}$ is changed to $R' = R \cup \{a_{j'}\} \setminus \{x\}$. The pattern is repeated until agent 1 gets a_{c-1} at his c-th pick. At this point $R' = R \cup \{a_c\} \setminus \{x\}$ for some $x \notin \{b, a_1, \ldots, a_c\}$.

If in each subsequent turn, agents other than 1 do not pick some item from $a_c, \ldots, a_{k-1}$, then we already have an RS-improvement for agent 1. Now assume for contradiction that at some time step $i > t_2$, some agent j other than 1 picks an item a_d where $c \leq d \leq k-1$. The reason j picks up a_d is because it is available in process $SA(\succ_1^*, \succ_{-1})$ but not available at that time step in $SA(\succ_1', \succ_{-1})$. At this time step j picks up x in $SA(\succ_1^*, \succ_{-1})$ which implies that $R' = R$. From now on, the sequential allocation process is identical in $SA(\succ_1', \succ_{-1})$ and $SA(\succ_1^*, \succ_{-1})$ because the set of remaining items is the same for both settings at exactly the same time step. Hence 1 picks the same subsequent items via $\succ_1'$ as via $\succ_1^*$. □

Example 9.

$$\succ_1: \quad o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}, o_{11}, o_{12}, o_{13}$$
$$\succ_2: \quad o_2, o_5, o_7, o_{11}, o_3, o_4, o_6, o_8, o_9, o_{10}, o_{12}, o_{13}, o_1$$
$$\succ_3: \quad o_3, o_{12}, o_8, o_{10}, o_1, o_2, o_4, o_5, o_6, o_7, o_9, o_{11}, o_{13}$$
$$\succ_1': \quad o_2, o_1, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}, o_{11}, o_{12}, o_{13}$$
$$\succ_1^*: \quad o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}, o_{11}, o_{12}, o_{13}$$

We denote profile $(\succ_1, \succ_2, \succ_3)$ by $\succ$ and we denote $(\succ_1', \succ_2, \succ_3)$ by $\succ'$. The allocation wrt truthful and misreported preferences are shown in Table 15.

Stage	$t_1 = 1$	$t_2 = 2$	3	4	5	6	7	8	9	10	$t_3 = 11$	12	13
Agent	1	2	3	1	2	3	1	2	3	1	2	3	1
Item picked in $\succ'$	o_2	o_5	o_3	o_1	o_7	o_{12}	o_4	o_{11}	o_8	o_6	o_9	o_{10}	o_{13}
Item picked in $\succ = \succ^*$	o_1	o_2	o_3	o_4	o_5	o_{12}	o_6	o_7	o_8	o_9	o_{11}	o_{10}	o_{13}

Table 15: Sequential allocation in which agent 1 can manipulate to get an improvement wrt responsive preferences over the truthful outcome. Agent 1's truthful report is $\succ_1$; $\succ_1^*$ is a report that may be different from $\succ_1$ but yields the same allocation for agent 1 as $\succ_1$. Report $\succ_1'$ gives a responsively better outcome over the truthful outcome.