Positional Scoring Rules for the Allocation of Indivisible Goods

Dorothea Baumeister[1], Sylvain Bouveret[2], Jérôme Lang[3], Trung Thanh Nguyen[1], Jörg Rothe[2], and Abdallah Saffidine[3]

[1] Heinrich-Heine Universität Düsseldorf
[2] LIG – Grenoble INP
[3] LAMSADE – Université Paris Dauphine

Abstract. We define a family of rules for dividing m indivisible goods among agents, parameterized by a scoring vector and a social welfare aggregation function. We assume that agents' preferences over sets of goods are additive, but that the input is ordinal: each agent simply ranks single goods. Similarly to positional scoring voting rules in voting, a scoring vector $s = (s_1, \ldots, s_m)$ consists of m nonincreasing nonnegative weights, where s_i is the score of a good assigned to an agent who ranks it in position i. The global score of an allocation for an agent is the sum of the scores of the goods assigned to her. The social welfare of an allocation is the aggregation of the scores of all agents, for some aggregation function $\star$ such as, typically, $+$ or min. The rule associated with s and $\star$ maps a profile of individual rankings over goods to (one of) the allocation(s) maximizing social welfare. After defining this family of rules and discussing some of their properties, we focus on the computation and approximation of winning allocations.

1 Introduction

Fair division of a divisible good has put forth an important literature about *specific procedures*, either centralized [11] or decentralized [5]. Fair division of *a set of indivisible goods* has, perhaps surprisingly, been mainly addressed by looking for allocations that satisfy a series of properties (such as equity or envy-freeness) and less often by defining specific allocation rules. A notable exception is a series of works that assume that each agent values each good by a positive number, the utility of an agent is the sum of the values of the goods assigned to her, and the resulting allocation maximizes social welfare; in particular, the *Santa Claus problem* [1] considers egalitarian social welfare, which maximizes the utility of the least happy agent. A problem with these rules is that they strongly rely on the assumption that the input is numerical. Now, as widely discussed in social choice, numerical inputs have the strong disadvantage that they suppose that interpersonal preferences are comparable. Moreover, from a practical designer point of view, eliciting numerical preferences is not easy: in contexts where money does not play any role, agents often feel more at ease expressing rankings than numerical utilities.

These are the main reasons why social choice – at least its subfield focusing on voting – usually assumes that preferences are expressed ordinally. Surprisingly, while voting rules defined from ordinal preferences have been addressed in hundreds of research articles, we can find only a few such works in fair division (with the notable exception of matching, discussed below). Brams, Edelman, and Fishburn [3] assume that agents rank single goods and have additively separable preferences; they define a Borda-optimal allocation to be one maximizing egalitarian social welfare, where the utility of an agent is the sum of the Borda scores of the objects she receives, and where the Borda score of object g_i for agent j ranges from 1 (for j's least preferred object) to m (for j's most preferred object). Unlike Brams *et al.* [3], Herreiner and Puppe [9] assume that agents should express rankings over *subsets* of goods, which, in the worst case, requires agents to express an exponentially large input, which should be avoided for obvious reasons.

One setting where it is common to use ordinal inputs is *two-sided matching*. But there, only one item is assigned to each agent, making this a rather different problem: fair division rules defined from ordinal inputs can be seen as a one-to-many extension of matching mechanisms. Examples of practical situations when one has to assign not a single, but several (sometimes many) items to each agent are common, and the expression of quantitative utilities is not always feasible: composition of sport teams, divorce settlements, exploitation of Earth observation satellites (see [5] for more examples).

We start by generalizing Borda-optimal allocations [3] to arbitrary scoring vectors and aggregation functions. Beyond Borda, the scoring vectors we consider are k-approval (the first k objects get score 1 and all others get 0), lexicographicity (an item ranked in position k counts more than the sum of all objects ranked in positions $k+1$ to n), and quasi-indifference (for short, QI: all objects have roughly the same score, up to small differences). As for aggregation functions, we focus on utilitarianism ($\star = +$) and egalitarianism ($\star = \min$, as well as $\star =$ leximin, which in a *strict sense* is not an aggregation function). In Section 2, we define these allocation rules and discuss some of their properties. Section 3 is devoted to the complexity of winner determination for a few combinations of a scoring vector and an aggregation function. In Section 4, we give several approximation results, some of which make use of *picking sequences*. Section 5 discusses some open questions for future research.

2 Positional Scoring Allocation Rules and Basic Properties

Let $N = \{1, \ldots, n\}$ be a set of agents and $G = \{g_1, \ldots, g_m\}$ a set of indivisible goods (we will use the terms *good* and *object* synonymously). An *allocation* is a partition $\pi = (\pi_1, \ldots, \pi_n)$, where $\pi_i \subseteq G$ is the bundle of goods assigned to agent i. In the general case, to compute an optimal allocation (for some notion of optimality) we would need, for every agent, her ranking over all subsets of G. As listing all (or a significant part of) the subsets of G would be infeasible in practice, we now make a crucial assumption: *agents rank only single objects*. This assumption is not without loss of generality, and has important consequences; in particular, it will not be possible for agents to express preferential dependencies between objects. Under this assumption, a *singleton-based profile* $P = (>_1, \ldots, >_n)$ is a collection of n rankings over G, and a *(singleton-based) allocation rule* (respectively, an *allocation correspondence*) maps any profile to an allocation (respectively, a nonempty subset of allocations).

We now define a family of allocation rules that more or less corresponds to the family of positional scoring rules in voting (see, e.g., [4]).

Definition 1. *A* scoring vector *is a vector* $s = (s_1, \ldots, s_m)$ *of real numbers such that* $s_1 \geq \cdots \geq s_m \geq 0$ *and* $s_1 > 0$. *Given a preference ranking* $>$ *over G and* $g \in G$, *let* $rank(g, >)$ *denote the* rank of g under $>$. *The* utility function over 2^G induced by the ranking $>$ on G and the scoring vector s *is for each bundle* $X \subseteq G$ *defined by* $u_{>,s}(X) = \sum_{g \in X} s_{rank(g,>)}$. *We consider the following specific scoring vectors:*

– *Borda scoring:* $\text{borda} = (m, m-1, \ldots, 1)$,[4]
– *lexicographic scoring:* $\text{lex} = (2^{m-1}, 2^{m-2}, \ldots, 1)$,
– *quasi-indifference for some* ε, $0 < \varepsilon \ll 1$*:*
  $\varepsilon\text{-qi} = (1 + (m-1)\varepsilon, 1 + (m-2)\varepsilon, \ldots, 1)$.
– *k-approval:* $k\text{-app} = (1, \ldots, 1, 0, \ldots, 0)$*, where the first k entries are ones and all remaining entries are zero.*

For example, let $G = \{a, b, c\}$ be a set of three goods and let two agents have the following preference profile: $(a >_1 b >_1 c,\ b >_2 c >_2 a)$. Let $\pi = (\{a\}, \{b, c\})$. Then, for the Borda scoring vector, agent 1's bundle $\{a\}$ has value 3 and agent 2's bundle $\{b, c\}$ has value $3 + 2 = 5$.

It is important to note that *we do not claim that these numbers actually concide, or are even close to, the agents' actual utilities* (although, in some specific domains, scoring vectors could be learned from experimental data). But this is the price to pay for defining rules from an *ordinal* input (see the Introduction for the benefits of ordinal inputs). This tradeoff is very common in voting theory: the well-studied family of *positional scoring rules* in voting theory (including the Borda rule) proceeds exactly the same way; voters rank alternatives, and the ranks are then mapped to scores; the winning alternative is the one that maximizes the sum of scores. If we aim at maximizing actual social welfare then we have to elicit the voters' (numerical) utilities rather than just asking them to rank objects. Caragiannis and Procaccia [6] analyze this ordinal-cardinal tradeoff in voting and show that the induced distortion is generally quite low.

The individual utilities are then aggregated using a monotonic, symmetric *aggregation function* that is then to be maximized. The three we will use here are among the most obvious ones: *sum* (utilitarianism), *min* and *leximin* (two versions of egalitarianism). Leximin refers to the (strict) lexicographic preorder over utility vectors whose components have been preordered nondecreasingly. Formally, for $x = (x_1, \ldots, x_n)$, let $x' = (x'_1, \ldots, x'_n)$ denote some vector that results from x by rearranging the components of x nondecreasingly, and define $x <_{\text{leximin}} y$ if and only if there is some i, $0 \leq i < n$, such that $x'_j = y'_j$ for all j, $1 \leq j \leq i$, and $x'_{i+1} < y'_{i+1}$, and $x \leq_{\text{leximin}} y$ means $x <_{\text{leximin}} y$ or $x = y$. Let leximin denote the maximum on a set of utility vectors according to $\leq_{\text{leximin}}$. For each scoring vector s, define three *allocation correspondences*:

– $F_{s,+}(P) = \text{argmax}_\pi \sum_{1 \leq i \leq n} u_{>_i,s}(\pi_i)$,
– $F_{s,\min}(P) = \text{argmax}_\pi \min_{1 \leq i \leq n} \{u_{>_i,s}(\pi_i)\}$, and
– $F_{s,\text{leximin}}(P) = \text{argleximin}_\pi (u_{>_1,s}(\pi_1), \ldots, u_{>_n,s}(\pi_n))$,

[4] Note that the usual definition of the Borda scoring vector in voting is $(m-1, m-2, \ldots, 1, 0)$. Here, together with [3] we fix the score of the bottom-rank object to 1, meaning that getting it is better than nothing. For scoring voting rules, a translation of the scoring vector has obviously no impact on the winner(s); for allocation rules, however, it does. See Example 1.

where $P = (>_1, \ldots, >_n)$ is a profile and $\pi = (\pi_1, \ldots, \pi_n)$ an allocation. Whenever we write $F_{s,\star}$, we mean any one of $F_{s,+}$, $F_{s,\min}$, and $F_{s,\text{leximin}}$. Similarly as in voting theory, an *allocation rule* is defined as the composition of an allocation correspondence and a tie-breaking mechanism (that break ties between allocations).

Example 1. For $n = 3$ agents and $m = 4$ goods, $G = \{a,b,c,d\}$, let $P = (c >_1 b >_1 a >_1 d,\ c >_2 a >_2 b >_2 d,\ b >_3 d >_3 c >_3 a) = (cbad,\ cabd,\ bdca)$. (From now on, we sometimes omit stating "$>_i$" explicitly in the preferences.) Then, $F_{(4,3,2,1),\text{leximin}}(P) = \{(c,ad,b)\}$ and $F_{(3,2,1,0),\text{leximin}}(P) = \{(c,a,bd)\}$.

Some properties of scoring voting rules naturally carry over to the scoring allocation rules. We here omit stating them explicitly and formally, except for *monotonicity* (Definition 2). Analogously to monotonicity of social welfare functions, monotonicity of an allocation rule means that no agent will ever lose a good by ranking it higher.

Definition 2. *An allocation rule $F_{s,\star}$ is* monotonic *if for all $\pi = (\pi_1, \ldots, \pi_n) \in F_{s,\star}(P)$ with $g \in \pi_i$ and for all profiles P' resulting from P by agent i ranking g higher, leaving everything else (i.e., the relative ranks of all other objects in i's ranking and the rankings of all other agents) unchanged, there exists some $\pi' = (\pi'_1, \ldots, \pi'_n) \in F_{s,\star}(P')$ with $g \in \pi'_i$.*

Proposition 1. *$F_{s,\star}$ is monotonic for every scoring vector s.*

Proof. For notational convenience, we give the proof only for $\star = +$. Let $P = (>_1, \ldots, >_n)$ be a profile over a set G of goods with $g \in G$ and let $P' = (>'_1, >_2, \ldots, >_n)$ be a modified profile, where w.l.o.g. the first agent modifies her preferences such that g is ranked higher in $>'_1$ than in $>_1$, leaving everything else unchanged. Let $\pi = (\pi_1, \ldots, \pi_n) \in F_{s,+}(P)$ be an allocation assigning good g to agent 1.

Fix an arbitrary $\pi' = (\pi'_1, \ldots, \pi'_n) \in F_{s,+}(P')$. For a contradiction, suppose that $g \notin \pi'_1$. For every good $g' \neq g$, the rank of g' in $>'_1$ is either the same as or below the rank of g' in $>_1$, and since $g \notin \pi'_1$, we have $u_{>'_1,s}(\pi'_1) \leq u_{>_1,s}(\pi'_1)$. By monotonicity of utilitarian aggregation, this implies

$$u'(\pi') = u_{>'_1,s}(\pi'_1) + \sum_{i=2}^{n} u_{>_i,s}(\pi'_i) \leq \sum_{i=1}^{n} u_{>_i,s}(\pi'_i) = u(\pi'). \tag{1}$$

Now, because $>'_1$ has been obtained by moving g upwards in $>_1$, we have $u_{>_1,s}(\pi_1) \leq u_{>'_1,s}(\pi_1)$. By monotonicity of utilitarian aggregation, this implies

$$u'(\pi) = u_{>'_1,s}(\pi_1) + \sum_{i=2}^{n} u_{>_i,s}(\pi_i) \geq \sum_{i=1}^{n} u_{>_i,s}(\pi_i) = u(\pi). \tag{2}$$

Since $\pi \in F_{s,+}(P)$ and $\pi' \in F_{s,+}(P')$, we have $u(\pi) \geq u(\pi')$ and $u'(\pi') \geq u'(\pi)$, which together with (1) and (2) implies $u'(\pi) \geq u(\pi) \geq u(\pi') \geq u'(\pi') \geq u'(\pi)$. Thus $u'(\pi) = u'(\pi')$, which means that $\pi \in F_{s,+}(P')$. But since $g \in \pi_1$, this is a contradiction, so the assumption is false and there is some $\pi' \in F_{s,+}(P')$ assigning good g to agent 1. ❑

Note that this result does *not* mean that $\pi \in F_{s,\star}(P)$ implies $\pi \in F_{s,\star}(P')$; e.g., this stronger property does not hold for the specific case of $\star = +$ and Borda scoring.

3 Winner Determination

In this section, we study the question: What is the complexity of determining an optimal allocation for a given scoring vector and a given aggregation function? For a given scoring vector s and a given aggregation function $F_{s,\star}$, where $\star \in \{+, \min, \text{leximin}\}$, define the following problem concerning winner determination.

$F_{s,\star}$-OPTIMAL-ALLOCATION ($F_{s,\star}$-OA)	
Given:	A profile P of n agents' rankings on a set G of indivisible goods and an allocation π of G.
Question:	Is π in $F_{s,\star}(P)$?

It is easy to see that $F_{s,+}$-OA is in P and both $F_{s,\min}$-OA and $F_{s,\text{leximin}}$-OA are in coNP for every scoring vector s.

The search problem $F_{s,\star}$-FIND-OPTIMAL-ALLOCATION ($F_{s,\star}$-FOA) seeks to actually *find* an optimal allocation. Clearly, $F_{s,+}$-FOA is solvable in polynomial time for any scoring vector s: every good is simply given to an agent who ranks it best. $F_{s,\min}$-FOA and $F_{s,\text{leximin}}$-FOA are much less easy in general.[5] We have the following easy polynomial-time upper bounds for restricted variants.

Proposition 2. *(1) For each k,* $F_{k\text{-app},\min}$-FOA *is solvable in polynomial time. (2)* $F_{s,\min}$-FOA *and* $F_{s,\text{leximin}}$-FOA *are solvable in polynomial time for every scoring vector s if there are a constant number of goods.*

(1) is a special case of the problem of maximizing egalitarian social welfare with a $\{0,1\}$-additive function, known to be solvable in polynomial time by applying a network flow algorithm [8]. In addition, we will study the following decision problem associated with the value of an optimal allocation.

$F_{s,+}$-OPTIMAL-ALLOCATION-VALUE ($F_{s,+}$-OAV)	
Given:	A profile $P = (>_1, \dots, >_n)$ of n agents' rankings on a set G of indivisible goods and $k \in \mathbb{N}$.
Question:	Is there an allocation $\pi = (\pi_1, \dots, \pi_n)$ such that $\sum_{1 \le i \le n} u_{>_i,s}(\pi_i) \ge k$?

Analogously, we define $F_{s,\min}$-OAV by asking whether or not $\min_{1 \le i \le n} u_{>_i,s}(\pi_i) \ge k$, and $F_{s,\text{leximin}}$-OAV where the bound is an ordered list $(k_1, \dots, k_n)$ of nonnegative integers and we ask whether $(u_{>_1,s}(\pi_1), \dots, u_{>_n,s}(\pi_n)) \ge_{\text{leximin}} (k_1, \dots, k_n)$.

Clearly, $F_{s,+}$-OAV is in P. Since the value of a given allocation for min and leximin can be computed in polynomial time, $F_{s,\min}$-OAV and $F_{s,\text{leximin}}$-OAV are in NP for each scoring rule s. For lexicographic scoring and quasi-indifference, these bounds are tight.

Theorem 1. $F_{\text{lex},\min}$-OAV *and* $F_{\text{lex},\text{leximin}}$-OAV *both are* NP*-complete.*

[5] Clearly, if the scoring vector s *is part of the input* then the problem $F_{s,\star}$-FOA is (weakly) NP-hard, even for two agents having the same preferences, by a direct reduction from PARTITION.

Proof. We only give the proof for $F_{\text{lex,min}}$-OAV (since it can be easily adapted to work for $F_{\text{lex,leximin}}$-OAV as well), by a reduction from the NP-complete problem EXACT-COVER-BY-3-SETS (X3C): given a collection $\mathscr{C} = \{C_1, \dots, C_p\}$ of 3-element subsets of a set X of size $3q$ (where $q < p$), is there an *exact cover of* X, *i.e.*, is there a subcollection $\mathscr{C}' \subset \mathscr{C}$ of size q such that each element of X appears in exactly one member of $\mathscr{C}'$?

From a given instance $(X, \mathscr{C})$ of X3C, with $\mathscr{C} = \{C_1, \dots, C_p\}$ a collection of 3-element subsets of X as above, we create an instance of the allocation problem as follows. We create one good g_i out of each element x_i from X, and a set $F = \{f_1, \dots, f_{p-q}\}$ of $p - q$ goods (to be called "first"), which makes a total of $2q + p$ goods. We create a set $\{1, \dots, p\}$ of p agents. Agent i has the following preferences: $f_1 >_i \cdots >_i f_{p-q} >_i C_i >_i X \setminus C_i$, where a set S in this order stands for all the goods of S in any fixed order.[6]

We claim that $(X, \mathscr{C})$ is a positive instance of X3C if and only if its constructed $F_{\text{lex,min}}$-OAV instance has an allocation with an egalitarian collective utility greater than or equal to $2^{3q-1} + 2^{3q-2} + 2^{3q-3}$ under lexicographic scoring.

($\Rightarrow$) Suppose that $\mathscr{C}$ is a positive instance of X3C and let $\mathscr{C}'$ be the corresponding exact cover of X. Let π be an allocation that gives to each agent i the goods corresponding to C_i if $C_i \in \mathscr{C}'$, and one good from F otherwise. Such an allocation π exists, since (i) the elements in $\mathscr{C}'$ do not overlap, and (ii) there are exactly $p - q$ agents i such that $C_i \notin \mathscr{C}'$ (and hence each such agent can receive a different $f_k \in F$). It is easy to see that each agent receiving one good amongst F has a utility greater than 2^{3q}, and each agent receiving one C_i has a utility equal to $2^{3q-1} + 2^{3q-2} + 2^{3q-3}$.

($\Leftarrow$) Let π be an allocation of egalitarian utility at least $2^{3q-1} + 2^{3q-2} + 2^{3q-3}$. Since $\|F\| = p - q$, at least q agents (call them "unhappy") do not receive any good from F. Suppose an unhappy agent i receives only a proper subset of the goods from C_i. Then the greatest utility she can get is $2^{3q-1} + 2^{3q-2} + 2^{3q-3} - 1$, if she gets her two preferred goods from C_i and all the goods from $X \setminus C_i$. Hence, for the egalitarian utility to be at least $2^{3q-1} + 2^{3q-2} + 2^{3q-3}$, each unhappy agent must get at least all the goods from C_i. Since the agents' shares cannot overlap, there can only be q unhappy agents, and their shares correspond to an exact cover of X.

Since this reduction can be computed in polynomial time, the proof is complete. ❑

Theorem 2. *For each fixed* ε, $0 < \varepsilon \ll 1$, $F_{\varepsilon\text{-qi,min}}$-OAV *and* $F_{\varepsilon\text{-qi,leximin}}$-OAV *both are* NP-*complete.*

Proof. Once again, we only give the proof for $F_{\varepsilon\text{-qi,min}}$-OAV, as its adaptation to $F_{\varepsilon\text{-qi,leximin}}$-OAV is easy. The proof is again by a reduction from the NP-complete problem X3C. Given an instance $(X, \mathscr{C})$ with $\mathscr{C} = \{C_1, \dots, C_p\}$ and $\|X\| = 3q$, create the following $F_{\varepsilon\text{-qi,min}}$-OAV instance. The set of objects is $G = \{g_1, \dots, g_{3q}\} \cup D$, where $D = \{d_1, \dots, d_{4(p-q)}\}$ is a set of dummy objects, hence $\|G\| = 4p - q$. There are p agents, where each agent i, $1 \le i \le p$, has the preference $C_i > X \setminus C_i > D$, and the bound is $k = 3 + (12p - 3q - 6)\varepsilon$.

($\Rightarrow$) Suppose that $(X, \mathscr{C})$ is a positive instance of X3C and let $\mathscr{C}'$ be an exact cover of X. Let π be an allocation that gives to each agent i the goods corresponding to C_i if

[6] Here and later, we slightly abuse notation, as X and C_i will refer both to the initial sets and their corresponding sets of goods.

$C_i \in \mathscr{C}'$, and otherwise four arbitrary goods from D that are still available. So π is such that $p-q$ agents receive four goods (and thus have a utility greater than 4), and q agents receive their three best goods, and hence they all have a utility of $3+(12p-3q-6)\varepsilon$.

($\Leftarrow$) Let π be an allocation of egalitarian utility at least $3+(12p-3q-6)\varepsilon$. By definition of QI, all agents must get at least three goods. Moreover, given the number of agents and goods, at least q "unhappy" agents must get exactly three goods (where "unhappy" is defined as in the proof of Theorem 1). Finally, given the bound, these unhappy agents must all get their three preferred goods, that is, C_i for agent i. Hence, all the C_i for the q unhappy agents must not overlap: this is an exact cover for $(X, \mathscr{C})$. ❑

An anonymous reviewer of a previous draft of this paper obtained the following result, and we are very grateful for his or her consent to include the proof.

Theorem 3. *$F_{\text{borda,min}}$-OAV and $F_{\text{borda,leximin}}$-OAV both are NP-complete.*

Proof. The construction to show NP-hardness is highly similar to the ones presented above (and once again we only give the proof for $F_{\text{borda,min}}$-OAV whose extension to $F_{\text{borda,leximin}}$-OAV is easy). Again, let $(X, \mathscr{C})$ be a given X3C instance with $\mathscr{C} = \{C_1, \ldots, C_p\}$ and $\|X\| = 3q$. Pad the X3C instance so that $3q-4 = 2(p-q)$ (this is similar to the padding employed by Faliszewski and Hemaspaandra [7]). There will be p agents, one for each subset C_i. Create $3q$ goods, one per element in X. These goods will be at the bottom of everyone's ranking; agent i's preferred goods among these $3q$ goods are the three goods corresponding to set C_i. There are another $2(p-q)$ goods and all the agents agree on their ranking. Now, either an agent receives set C_i with value $9q-3$, or she receives two higher valued goods with values $6q-3-i$ and $3q+i$, or $9q-3$ in total.

In a bit more details, for showing that we have a yes-instance of X3C if and only if the constructed instance of $F_{\text{borda,min}}$-OAV is a yes-instance, the necessary part is easy to see. For the sufficient part, note that to obtain a utility of at least $9q-3$ for each of the p agents, everyone needs to get at least two goods. Since there are $3q+2(p-q) = q+2p$ goods, there are at least $p-q$ agents that receive only two goods. To obtain a utility of at least $9q-3$ with only two goods, the lower ranked good must be placed in the first $2(p-q)$ positions. This implies that the $p-q$ agents receiving only two goods receive those $2(p-q)$ goods that are placed at the beginning of every preference. Then the remaining q agents must all receive the goods at positions $3q$, $3q+1$, and $3q+2$ to obtain a utility of $9q-3$, and this corresponds to an exact cover of X. ❑

Using a slight adaptation of the proofs of Theorems 1, 2 and 3, we can show that $F_{\text{lex,min}}$-OA, $F_{\varepsilon\text{-qi,min}}$-OA and $F_{\text{borda,min}}$-OA are coNP-complete. These proofs, however, do not directly extend to $F_{\text{lex,leximin}}$-OA, $F_{\varepsilon\text{-qi,leximin}}$-OA nor $F_{\text{borda,leximin}}$-OA.

Proposition 3. *For $s \in \{\text{borda}, \text{lex}, \varepsilon\text{-qi}\}$, $F_{s,\text{min}}$-OA is coNP-complete.*

Proof Sketch. For $s = \text{lex}$, we can use a reduction from a restricted version of the complementary of X3C, which we will call R-$\overline{\text{X3C}}$ and define as follows: given a triple $(X, \mathscr{C}, \mathscr{C}')$, where $(X, \mathscr{C})$ is an instance of X3C, and $\mathscr{C} = \{C_1, \ldots, C_p\}$ and $\mathscr{C}' = \{C'_1, \ldots, C'_q\}$ are such that (i) for all i, $C'_i \subset C_i$, (ii) for all i, $\|C'_i\| = 2$ and (iii) for all $i \neq j$, $C'_i \cap C'_j = \emptyset$, is $(X, \mathscr{C})$ a negative instance of X3C? This problem can be proven to be coNP-complete by using a reduction from $\overline{\text{X3C}}$. Suppose, w.l.o.g., that

$X \setminus \bigcup_{i=1}^{q} C'_i = \{x_1, \ldots, x_q\}$. We adapt the reduction used in Theorem 1 by constraining the preferences of the first q agents as follows: for each $i \in \{1, \ldots, q\}$,

(i) among the three objects from C_i, we put those from C'_i at the first two positions and
(ii) among the objects from $X \setminus C_i$, we put g_i at the first position.

Now let π be as follows: each $i \in \{1, \ldots, q\}$ gets the two objects from C'_i and x_i and each agent $i \in \{q+1, \ldots, p\}$ gets f_{i-q}. We can prove that $(X, \mathscr{C}, \mathscr{C}')$ is a positive instance of R-$\overline{\text{X3C}}$ if and only if $\pi \in F_{s,\min}(P)$.

The proofs for $s = \varepsilon$-qi and $s =$ borda also use a reduction from R-$\overline{\text{X3C}}$, using the same kind of adaptation of the proof of Theorems 2 and 3 as above for Theorem 1. For $s = \varepsilon$-qi (resp. $s =$ borda), the allocation π gives the two objects from C'_i and x_i to each agent $i \in \{1, \ldots, q\}$ and four random objects d_i (resp. 2 objects among the $2(p-q)$ first ones) to each agent $i \in \{q+1, \ldots, p\}$. ❑

For a constant number of agents, we provide efficient algorithms for many of our problems via dynamic programming.

Theorem 4. *For each* $s \in \{\text{borda}, \text{lex}, \varepsilon\text{-qi}\}$ *and for each* $\star \in \{\min, \text{leximin}\}$, $F_{s,\star}$-OA *and* $F_{s,\star}$-FOA *are solvable in polynomial time if the number of agents is constant.*

Proof Sketch. For $s \in \{\text{borda}, \varepsilon\text{-qi}\}$, we sketch an algorithm that works for both $F_{s,\star}$-OA and $F_{s,\star}$-FOA. It encodes each possible allocation that assigns the first j goods to the n agents as an n-dimensional vector. After initializing $V_0 = \{0\}$, it runs in m steps. At step j, it generates the vector set V_j from V_{j-1} by putting, for each $v \in V_{j-1}$, n vectors $v_i = v + s_{rank(g_j, >_i)} \cdot e_i$, $1 \leq i \leq n$, into V_j, where e_i denotes the i-th unit vector. Clearly, $\|V_j\| \leq \|V_m\|$ for all $j \leq m$. For $s =$ borda, every entry of each vector in V_m is bounded above by $m(m+1)/2$ and thus $\|V_m\| \in O(m^{2n})$. For $s = \varepsilon$-qi, every entry of each vector in V_m has the form $p + q \cdot \varepsilon$, where $p, q \in \mathbb{Z}$, $0 \leq p \leq m$ and $0 \leq q \leq m(m-1)/2$. Hence, $\|V_m\| \in O(m^{3n})$. It is not difficult to see that the running time of the algorithm depends on $\|V_m\|$ and thus is polynomial in m.

We now turn to lexicographic scoring, $s =$ lex. The case with two agents can be solved efficiently by implementing the following simple rules when the preferences of the agents are examined from the most to the least preferred good: (1) If the agents have different goods that are not assigned yet on the current position, both agents get their current goods and proceed with the next position. (2) If both current objects are already assigned, proceed with the next position. (3) If both agents rank the same object, say g, that is not assigned yet on the current position, then let $g_i \neq g$ be the most preferred good of agent $i \in \{1,2\}$ that has not been assigned yet, and w.l.o.g. assume that $rank(g_1, >_1) \geq rank(g_2, >_2)$. Assign g to agent 1 and all remaining objects to agent 2. (4) The last case is that only one of the current objects, say g (w.l.o.g., the one ranked by agent 1), has not been assigned yet. If g is not the most preferred good of agent 2 among those not yet assigned, then assign it to agent 1 and remaining objects to agent 2. Otherwise, let $g_i \neq g$ be the most preferred good of agent $i \in \{1,2\}$ that has not been assigned yet. Note that $rank(g, >_1) < rank(g, >_2)$. (i) If $rank(g_1, >_1) < rank(g, >_2)$, agent 1 receives g_1, while agent 2 gets g and the remaining objects. (ii) If $rank(g_1, >_1) = rank(g, >_2)$, assign g_1 to agent 1 and g to agent 2, and proceed with the next position. (iii) If $rank(g, >_2) < rank(g_1, >_1) < rank(g_2, >_2)$, assign g to agent 2 and the remaining objects to agent 1. (iv) If $rank(g_1, >_1) \geq rank(g_2, >_2)$, give g to agent 1 and the remaining objects to agent 2. The case with more than two agents can be proven by induction on n and is omitted due to space constraints. ❑

		OA	OAV	FOA
$F_{s,+}$		in P	in P	pol. time
$F_{s,\min}$		coNP-comp*	NP-comp*	NP-hard*
	k-app or $m \in O(1)$	in P	in P	pol. time
	lex or ε-qi	coNP-comp	NP-comp	NP-hard
	borda	coNP-comp	NP-comp	
	lex or borda or ε-qi, if $n \in O(1)$	in P	in P	pol. time
$F_{s,\text{leximin}}$		coNP-comp*	NP-comp*	NP-hard*
	lex or ε-qi	in coNP	NP-comp	NP-hard
	borda	in coNP	NP-comp	
	lex or borda or ε-qi, if $n \in O(1)$	in P	in P	pol. time

*if s is part of the input (even for two agents with same preferences)

Table 1. Overview of complexity results (gray: partial results)

4 Approximation

$F_{\text{lex},\min}$-OAV is NP-complete by Theorem 1. This raises the issue of whether there exists a polynomial-time approximation algorithm for the search variant of this rule; this turns out to be the case.

Proposition 4. *There exists a* $(1/2)$*-approximation algorithm for* $F_{\text{lex},\min}$-FOA.

Proof Sketch. Let k^* be the smallest integer such that it is possible to give each agent one of her best k^* objects. Finding an allocation that gives each agent one of her best i objects can be expressed as a bipartite matching problem, therefore k^* and a corresponding allocation π_{k^*} can be computed in polynomial time. We complete the partial allocation π_{k^*} into a complete allocation π in an arbitrary way. In π, each agent gets a utility at least 2^{m-k^*}, therefore the egalitarian social welfare of π is at least 2^{m-k^*}. Now, if there were an allocation with egalitarian social welfare at least 2^{m-k^*+1}, then it would have been possible to give each agent one of her best $k^* - 1$ objects, which would contradict the definition of k^*. ❑

We now turn to a different kind of approximation: *picking sequences*, whose advantage is that they avoid preference elicitation. We investigate the price to pay for that: in Section 4.1 (respectively, Section 4.2), we focus on the ratio (respectively, the difference) between the value of the optimal allocation and the value of the allocation obtained by applying a picking sequence.

4.1 Multiplicative Price of Elicitation-Freeness

Simple protocols for allocating indivisible resources without eliciting the agents' preferences first, as discussed in [5,2,10], consist in asking agents to pick objects one after the other, following a predefined sequence. An interesting question is whether using such protocols (without elicitation), or simulating them from the known preferences (after full elicitation of the agents' rankings) gives a good approximation of our scoring rules: what is the loss incurred by the application (simulated or not) of the picking sequence with respect to an optimal allocation? We give here two results for Borda scoring: one for egalitarianism, one for utilitarianism. One may wonder why we should look for such a result in the case of utilitarianism, given that there is a straightforward greedy algorithm that outputs an optimal allocation. The reason is that picking sequences (when actually used, as opposed to simulated ones) do better on one criterion: they are very cheap in communication, as agents only reveal part of their preferences by picking objects, as opposed to revealing their full preferences in the case of a centralized protocol.

Formally, a *(picking) policy* is a sequence $\sigma = \sigma_1 \cdots \sigma_m \in \{1,\ldots,n\}^m$, where at each step, agent σ_i picks her most preferred object among those remaining (where we assume agents to use only their sincere picking strategies). For instance, if $m = 4$ and $n = 2$, 1221 is the sequence where 1 picks an object first, then 2 picks two objects, and 1 takes the last object. The precise definition of an allocation induced by a picking sequence and a profile, assuming that agents act according to their true preferences, is in [2]. Sequential allocation rules are appealing because they require even less input from the agents than singleton-based allocation rules; however, this gain in communication comes with a loss of social welfare. To quantify this loss, we define the following measure.

Definition 3. *Given a policy σ (for n agents and m objects), a scoring vector s, and an aggregation function $\star \in \{+, \min\}$, the* multiplicative price of elicitation-freeness *of σ, denoted by $MPEF_{s,\star}(\sigma)$, is the worst-case ratio in social welfare between an optimal allocation for $F_{s,\star}$ and the sequential allocation, among all profiles with m goods.*

Since we focus on $s = \text{borda}$ only, we from now on simply write $MPEF_\star(\sigma)$ to mean $MPEF_{\text{borda},\star}(\sigma)$. We now give results about the quality of the outcome of *balanced picking sequences* $(12\cdots n)^{\frac{m}{n}}$, assuming that m is a multiple of n. For instance, if $m = 6$ and $n = 3$, $\sigma = 123123$ is balanced. Computing the price of elication-freeness is challenging. We focus on the *regular policy* $\sigma_{\text{R}}^n = (1\cdots n)^*$, but we can get similar results for other fair policies like $(1\cdots nn\cdots 1)^*$.

Lower Bounds A naive algorithm for computing the additive or multiplicative PEF for a given value m is simply to generate all possible profiles and for each of them to compute an optimal allocation from which it is possible to deduce the loss incurred by the sequential allocation. However, the number of profiles grows exponentially in m, and computing an optimal allocation might be intractable. Still, it is possible to lower-bound the PEF for a given m by computing the incurred loss for a subset of all possible profiles. In Figure 1, we plot the best such lower bounds we could achieve experimentally for the multiplicative PEF. In the case $\star = +$, each data point corresponds to two million profiles randomly generated (with a uniform distribution). In the case $\star = \min$, for each data point, random profiles were generated until a threshold of 1,800 seconds of computation time was reached. The conclusions that can be drawn from Figure 1 is that for $\star = +$, in the worst and average cases the loss seems to tend to the neighborhood of 1. The conclusions for $\star = \min$ are somewhat similar, but they are less firm, as we have not been able to go as far in the number of objects as for $\star = +$.

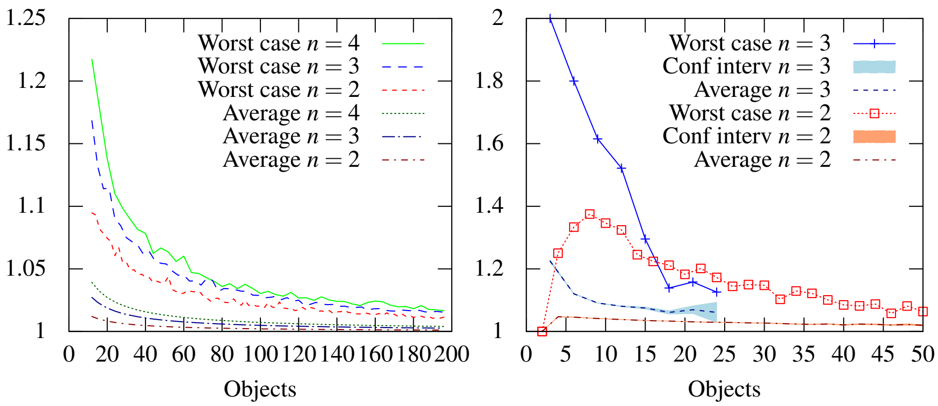

Fig. 1. *Left:* Multiplicative PEF for $\star = +$, Borda scoring, and regular policies σ_{R}^2, σ_{R}^3, σ_{R}^4. *Right:* Multiplicative PEF for $\star = \min$, Borda scoring, and regular policies σ_{R}^2 and σ_{R}^3.

We now provide a formal lower bound for *MPEF* for $\star = +$, and the regular policy.

Proposition 5. *For $m = kn$ objects, $MPEF_+(\sigma_R^n) \geq 1 + \frac{mn-m-n^2+n}{m^2+mn}$, and thus we have $MPEF_+(\sigma_R^n) \geq 1 + \frac{n-1}{m} + \Theta(1/m^2)$ when m tends to $+\infty$ with n being held constant.*

Proof Sketch. We construct a profile $P = (>_1, >_2, \ldots, >_n)$ where for each agent a, $>_a$ is defined so that (a) $\forall i, i'$ such that $i < i' \leq m - a$, $o_i > o_{i'}$; (b) $\forall j, j'$ such that $m - a + 1 \leq j < j'$, $o_j > o_{j'}$; (c) $\forall i \leq m - a, \forall j \geq m - a + 1$, $o_i < o_j$. For each $i \in [1, n]$, object o_{m-i+1} is assigned to agent i in the sequential and an optimal allocation. For each $i \in [1, m-n]$, object o_i is assigned to agent 1 in an optimal allocation with utility $m - i$. However, if $i \equiv a \pmod{n}$, then o_i is assigned to agent a in the sequential allocation with utility $m - i - (a-1)$. Summing over all objects leads to the result. ❑

Upper Bounds We now also provide formal upper bounds for *MPEF* for $\star = +$ and $\star = \min$, and the regular policy.

Proposition 6. *For $m = kn$ objects, $MPEF_+(\sigma_R^n) \leq 2 - \frac{m-n}{mn+n}$, and thus $MPEF_+(\sigma_R^n) \leq 2 - \frac{1}{n} + \Theta(1/m)$ when m tends to $+\infty$ with n being held constant.*

Proof Sketch. Let g_{ni+j} be the object picked at the $(ni+j)$th time step. Because σ is balanced, it is picked by agent j. Let $u_p(g)$ be the score associated to object g by player p: $u_p(g) = u_{>_p,s}(\{g\}) = s_{rank(g,>_p)}$. The loss of social welfare associated with σ is the sum of the losses over each object g_{ni+j}, which can be expressed as $\max_{0 \leq j' \leq n-1} u_{j'}(g_{ni+j}) - u_j(g_{ni+j})$.

At step $ni+j$, when it is agent j's turn to pick an object, the following facts hold: (a) no more than $ni+j-1$ objects have already been picked, so agent j will pick an object among her $ni+j$ best objects; (b) object g_{ni+j} hasn't been picked by any other agent so far; therefore, g_{ni+j} is not among the best i objects of any agent. (a) and (b) imply (a') $u_j(g_{ni+j}) \geq s_{ni+j} = m-(ni+j)+1$ and (b') $u_{j'}(g_{ni+j}) \leq s_i = m-i+1$. ¿From (a') and (b') we get that the ratio of social welfare associated with object g_{ni+j} is upper-bounded by $\frac{m-i+1}{m-(ni+j)+1}$. Summing over all objects leads to the result. ❑

Corollary 1. *If $n=2$ and $m=2k$, $1+\frac{m-2}{m(m+2)} \leq MPEF_{+}(\sigma_R^2) \leq \frac{3}{2}+\frac{3}{2m+2}$.*

Proposition 7. *For $m=kn$ objects, $MPEF_{\min}(\sigma_R^n) \leq \frac{2mn-m+n}{mn+2n-n^2}$, and thus $MPEF_{+}(\sigma_R^n) \leq 2-\frac{1}{n}+\Theta(1/m)$ when m tends to $+\infty$ with n being held constant.*

Proof. The best allocation one could hope for would give every agent her preferred k objects, and it has social welfare $\sum_{i=1}^{k}(m-i+1) = \sum_{i=1}^{k}(m+1) - \sum_{i=1}^{k} i$. The worst case occurs when all agents have the same preference; in this case, the least well-off agent is n, who gets the objects he ranked n, $2n$, ..., and kn, and his utility (and therefore the social welfare) is $\sum_{i=1}^{k} s_{ni} = \sum_{i=1}^{k}(m-ni+1) = \sum_{i=1}^{k}(m+1) - n\sum_{i=1}^{k} i$.

Therefore, skipping the intermediate computation steps, we have $MPEF_{\min}(\sigma) \leq \frac{\sum_{i=1}^{k}(m+1)-\sum_{i=1}^{k} i}{\sum_{i=1}^{k}(m+1)-n\sum_{i=1}^{k} i} = \frac{2k(m+1)-k(k+1)}{2k(m+1)-nk(k+1)} = 2-\frac{1}{n}+\frac{2n^2+2n-2}{mn+n^2+2n}$, which concludes the proof. ❑

Corollary 2. *If $n=2$ and $m=2k$, $MPEF_{min}(\sigma_R^2) \leq \frac{3}{2}+\frac{5}{m+4}$.*

4.2 Additive Price of Elicitation-Freeness

Definition 4. *Given a policy σ (for n agents and m objects), a scoring vector s, and an aggregation function $\star \in \{+, \min\}$, the* additive price of elicitation-freeness *of σ, denoted by $APEF_{s,\star}(\sigma)$, is the worst-case difference in social welfare between the sequential allocation and an optimal allocation for $F_{s,\star}$ among all profiles with m goods.*

Since we focus on $s=$ borda only, we simply write $APEF_{\star}(\sigma)$ to mean $APEF_{\text{borda},\star}(\sigma)$.

We now provide a formal lower bound linear in m for $\star=+$, with a fixed number of agents n and the regular policy.

Proposition 8. *For $m=kn$ objects, $APEF_{+}(\sigma_R^n) \geq \frac{(n-1)(m-n)}{2}$.*

Proof Sketch. We build a profile $P=(>_1,\ldots,>_n)$ where for each agent a, $>_a$ is defined so that (a) $\forall i,i'$ such that $i<i' \leq m-a$, $o_i > o_{i'}$; (b) $\forall j,j'$ such that $m-a+1 \leq j < j'$, $o_j > o_{j'}$; (c) $\forall i \leq m-a$, $\forall j \geq m-a+1$, $o_i > o_j$. For each $i \in [1,n]$, object o_{m-i+1} is assigned to agent i in the sequential and an optimal allocation. $\forall i \in [1,m-n]$, object o_i is assigned to agent 1 in an optimal allocation with utility $m-i$. However, if $i \equiv a \pmod{n}$, then o_i is assigned to agent a in the sequential allocation with utility $m-i-(a-1)$. The loss of social welfare is $a-1$ for each object o_i such that $i \in [1,m-n]$ and $i \equiv a \pmod{n}$. The loss of social welfare in profile P is thus $(\frac{m}{n}-1)\sum_{a=1}^{n}(a-1) = (n-1)(m-n)/2$. ❑

Figure 2 is the equivalent of Figure 1 for additive lower bounds. We can easily see in this figure that that for $\star=+$, in the worst case the loss seems to be in the order of m (which is good), whereas in the average case the loss seems to grow also linearly with m (similarly for $\star=\min$, with the same remarks as for multiplicative lower bounds).

We now also provide a formal upper bound quadratic in m with a fixed number of agents n, for $\star=+$ and $\star=\min$, and the regular policy.

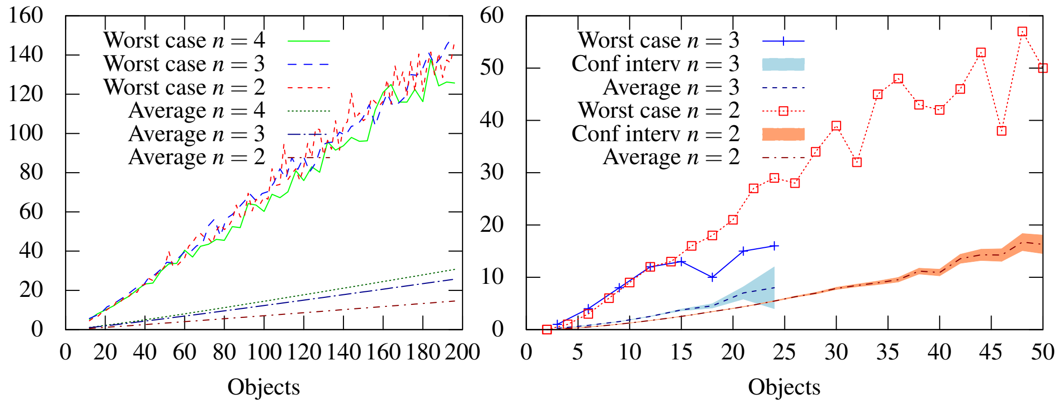

Fig. 2. *Left:* Additive PEF for $\star = +$, Borda scoring, and regular policies σ_R^2, σ_R^3, and σ_R^4. *Right:* Additive PEF for $\star = \min$, Borda scoring, and regular policies σ_R^2 and σ_R^3.

Proposition 9. *For $m = kn$ objects, $APEF_+(\sigma_R^n) \le \frac{(m-n)(mn-m+n^2+n)}{2n}$.*

Proof Sketch. Let g_{ni+j} be the objects picked at the $(ni+j)$th time step. Because σ_R^n is balanced, it is picked by agent j. Let $u_p(g)$ be the score associated to object g by player p: $u_p(g) = u_{>_p,s}(\{g\}) = s_{rank(g,>_p)}$. The loss of social welfare associated with σ_R^n is the sum of the losses over each object g_{ni+j}, which can be expressed as $\max_{0 \le j' \le n-1} u_{j'}(g_{ni+j}) - u_j(g_{ni+j})$.

At step $ni+j$, when it is agent j's turn to pick an object, the following facts hold: (a) no more than $ni+j-1$ objects have already been picked, so agent j will pick an object among her $ni+j$ best objects; (b) object g_{ni+j} hasn't been picked by any other agent so far; therefore, g_{ni+j} is not among the best i objects of any agent. (a) and (b) imply (a') $u_j(g_{ni+j}) \ge s_{ni+j}$ and (b') $u_{j'}(g_{ni+j}) \le s_{i+1}$. Finally, since s is the Borda scoring vector, we have $s_y - s_x = x - y$. From (a') and (b') we get that the loss of social welfare associated with object g_{ni+j} is upper-bounded by $s_i - s_{ni+j} = (n-1)i + j$. Summing over all objects leads to the desired result. ❑

Corollary 3. *For $n = 2$ and $m = 2k$, $\frac{m}{2} - 1 \le APEF_+(\sigma_R^2) \le \frac{m^2}{4} + m - 3$.*

Proposition 10. *For $m = kn$ objects, $APEF_{\min}(\sigma_R^n) \le \frac{m^2n - mn - m^2 + mn^2}{2n^2}$.*

Proof. The best allocation one could hope for would give every agent her preferred k objects, and it has social welfare $\sum_{i=1}^{k}(m-i+1) = \sum_{i=1}^{k}(m+1) - \sum_{i=1}^{k} i$. The worst case occurs when all agents have the same preferences; in this case, the worst-off agent is n, who gets the objects she ranked n, $2n$, $\ldots$, and kn, and her utility (and therefore the social welfare) is $\sum_{i=1}^{k} s_{n_i} = \sum_{i=1}^{k}(m - ni + 1) = \sum_{i=1}^{k}(m+1) - n\sum_{i=1}^{k} i$. Therefore, $APEF_{\min}(\sigma_R^n) \le \left(\sum_{i=1}^{k}(m+1) - \sum_{i=1}^{k} i\right) - \left(\sum_{i=1}^{k}(m+1) - n\sum_{i=1}^{k} i\right) = n\sum_{i=1}^{k} i - \sum_{i=1}^{k} i = (n-1)\frac{k(k+1)}{2} = (n-1)\frac{m(n+m)}{2n^2} = \frac{m^2n - nm - m^2 + mn^2}{2n^2}$, which concludes the proof. ❑

This upper bound is asymptotically better (by a factor of n) than the upper bound for $APEF_+(\sigma_R^n)$. In particular, for two agents, it is in the order of $m^2/8$ (to be compared with $m^2/4$ for $\star = +$ in Corollary 3).

5 Conclusions and Outlook

We have defined a general family of allocation rules for indivisible goods, which can be seen as the counterpart, for resource allocation, of positional scoring rules from voting theory. Each rule is parameterized by a scoring vector and an aggregation function. Focusing on four scoring vectors and three aggregation functions, we have determined the complexity of computing an optimal allocation for almost all rules considered here (see Table 1 for the list of results, and the problems whose precise complexity remains unknown). We have also given some approximation results, some of which make use of picking sequences whose main purpose it is to avoid preference elicitation.

Even if winner determination is computationally hard for many choices of s and $\star$ (except for the trivial case of $\star = +$), these rather negative results should be tempered by the fact that in most practical settings the number of agents and items is sufficiently small (*e.g* $n \leq 5$, $m \leq 20$) for the optimal allocation to be computed, even when its determination is NP-hard. Moreover, the results of Section 4 show that good approximations of optimal allocations can often be determined with a very low communication cost.

An issue that we did not consider here is manipulability. Clearly, almost all of our rules are manipulable; characterizing exactly the family of allocation rules that are manipulable and measuring the extent to which our rules are computationally resistant to manipulation is clearly an interesting topic for further research.

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