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Chore Division on a Graph

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Abstract The paper considers fair allocation of indivisible nondisposable items that generate disutility (chores). We assume that these items are placed in the vertices of a graph and each agent's share has to form a connected subgraph of this graph. Although a similar model has been investigated before for goods, we show that the goods and chores settings are inherently different. In particular, it is impossible to derive the solution of the chores instance from the solution of its naturally associated fair division instance. We consider three common fair division solution concepts, namely proportionality, envy-freeness and equitability, and two individual disutility aggregation functions: additive and maximum based. We show that deciding the existence of a fair allocation is hard even if the underlying graph is a path or a star. We also present some efficiently solvable special cases for these graph topologies.

Keywords Computational social choice · resource allocation · fair division · indivisible chores

CR Subject Classification J.4 (Economics) · I.2.11 (Multiagent Systems) · F.2.2 (Computations on discrete structures)

1 Introduction

Fair division of goods and resources is a practical problem in many situations and a popular research topic in Economics, Mathematics and Computer science. Sometimes, however, the objects that people have to deal with are undesirable, i.e., instead of utility create some cost. Imagine that a cleaning service firm allocates to its teams a set of offices, corridors, etc in a building. Each team has some idea of how much effort each room requires. The cost of

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the whole assignment for a team may be in the form of time the team will have to spend on the job, and this might depend on whether the work in all assigned rooms can start simultaneously or whether they have to be treated one after another. Moreover, for practical reasons, it is desirable that each team's assignment is a contiguous set of rooms. As another example consider a firm that supervises the operation of the computer network during a conference. Each of its employees has to choose one of the possible shifts, but shifts scheduled at different times of the conference may incur different opportunity cost for various persons. Moreover, we can assume that everybody prefers to have just one uninterrupted period to spend at work.

The constraints described above could be represented by an undirected graph whose vertices are rooms or time intervals and there is an edge between two vertices if the respective rooms are adjacent; or the corresponding time periods immediately follow each other. Each agent should obtain a connected piece of the underlying neighborhood graph. Complete graphs correspond to the 'classic' case with no connectivity constraints. By contrast, paths and stars represent the simplest combinatorial structures, and yet such graphs can be used to model a rich variety of situations. For example, a path may represents successive time intervals and a star corresponds to a building with a central foyer and mutually non-connected rooms accessible from this foyer.

Related work. The mathematical theory of fair division started with the seminal paper of Steinhaus (1948). Although originally most researchers focused on *divisible* items (the topic is also known as the *cake-cutting problem*), fair division of indivisible goods has also received a considerable amount of attention in the traditional and computational social choice literature. The interested reader can for instance read the survey by Bouveret et al. (2015) for an overview of this topic.

Several recent papers combine graphs and fair division of indivisible items by assuming that agents are located on the vertices of a graph. Abebe et al. (2017) and Bei et al. (2017) define an allocation to be locally envy-free if no agent envies a neighbor's allocation, and they call it locally proportional if each agent values her own allocation at least as much as the average value of allocations of her neighbors. The former work characterizes graphs for which a single-cutter protocol can give locally envy-free (and thus also locally-proportional allocations) and the latter proposes a moving-knife algorithm that outputs an envy-free allocation on trees. Bredereck et al. (2018) analyze the classic and the parameterized complexity of finding allocations that are locally envy-free and simultaneously complete, Pareto efficient, or optimize the utilitarian social welfare.Chevaleyre et al. (2007) and Gourvès et al. (2017) consider the case where each agent has an initial endowment of goods and can trade with her neighbors in the graph. The authors study outcomes that can be achieved by a sequence of mutually beneficial deals.

In the paper by Bouveret et al. (2017) for the first time a graph-based constraint on agents' bundles has been imposed. The authors model the items as vertices of a graph and they require that each agent should receive a bundle that forms a connected subgraph. They show that even if the underlying graph is a path the problems to decide whether there exist proportional or envy-free divisions with connected shares are NP-complete. In case of stars, envy-freeness is also intractable, but a proportional division can be found by a polynomial algorithm. In addition, Bouveret et al. (2017) study maximin share (MMS) allocations, a fairness notion introduced by Budish (2011). [1]They show that a MMS allocation always

[1] Consider all the possible divisions of the cake into n disjoint pieces. The maximim share of an agent is her utility for the worst piece taken in an allocation that is most favourable for her in this respect. A MMS allocation is one where each agent is ensured a piece with utility at least her maximin share.

exists if the underlying graph is a tree, provide a polynomial algorithm to find one in this case and show that a cycle may admit no MMS allocation. Lonc and Truszczynski (2018) study MMS allocations on cycles in greater depth. They identify several cases when MMS allocations always exist (*e.g.*, at most three agents and at most 8 goods, the number of goods not exceeding twice the number of agents, fixed number of agent types) and provide results on allocations guaranteeing each agent a certain portion of her maximin share.

Now we review some other works that also consider allocations on a graph with the additional constraint that each bundle has to be connected. First, Suksompong (2017) deal with paths only and approximately fair (proportional, envy-free and equitable) allocations up to an additive approximation factor. He shows that for all the three fairness notions there is a simple approximation guarantee derived from the maximum value of an item and that for proportionality, as well as equitability, an allocation achieving this bound can be computed efficiently.

Bilò et al. (2018) deal with two relaxations of envy-freeness: envy-free up to one good, briefly EF1 (an agent does not think that another agent's bundle, possibly with one of its outer items removed, is more valuable than her own bundle) and envy-free up to two outer goods, briefly EF2. They characterize graphs admitting allocations fulfilling these notions and provide efficient algorithms to find such allocations. Oh et al. (2018) present an algorithm that computes a contiguous EF1 allocation for three agents with identical valuations using a logarithmic number of queries.

Igarashi and Peters (2018) study Pareto-optimality. They show that for paths and stars a Pareto optimal allocation can be found efficiently, but the problem is NP-hard even for trees of bounded pathwidth. They also show that it is NP-hard to find a Pareto-optimal MMS allocation even on a path.

It is worth noting that although the study of connected fair division is relatively recent in the context of indivisible items, there is an important literature on the contiguity requirement in the context of cake-cutting. From the great number of various results we consider among the most interesting ones the contrast between the proven existence of envy-free (Stromquist 1980) and equitable (Aumann and Dombb 2010; Cechlárová et al. 2013) divisions with connected pieces and, on the other hand, the nonexistence of finite algorithms for computing them (Stromquist 2008; Cechlárová and Pillárová 2012).

Beyond considering the connectivity constraint, an important aspect in which our work departs from the mainstream literature on fair division is the fact that we consider negative items (chores). Chore division of divisible goods was mentioned for the first time by Gardner (1978). Although straightforward modifications of some algorithms for positive utilities can also be applied to the chore division context (*e.g.*, the Moving Knife algorithm for proportional divisions), it happens more often that chore division problems are more involved than their fair division counterparts. For example, the discrete algorithm for obtaining an envy-free division of divisible chores for three persons by Oskui (Robertson and Webb 1998, pages 73-75) needs nine cuts and the procedure based on using four moving knifes makes 8 cuts (Peterson and Su 2002), while in the Selfridge's algorithm for envy-free (positive) division (Woodall 1980) five cuts suffice. If the number of agents is 4, the moving-knife procedure by Brams et al. (1997) needs 11 cuts, while the first algorithm for envy-free division of chores, given by Peterson and Su (2002), needs 16 cuts.

The fact that chore division has been given much less attention in research is mirrored also in monographs on fair division. For example, Robertson and Webb (1998) only deal with chores in Section 5.5. Chapter 7 in the book *Economics and Computation* edited by Rothe (2015) deals with cake cutting, but only Section 7.4.6 treats chores. Chapter 12 on fair division of indivisible goods in the *Handbook of Computational Social* Choice by Bouveret et al. (2015) does not mention chores at all.

Of the more recent works on chore division let us mention Caragiannis et al. (2012) who deal with divisible and indivisible goods and chores from the point of view of the price of fairness for three fairness notions. Heydrich and van Stee (2015) consider the price of fairness for the fair division of infinitely divisible chores. Aziz et al. (2017) also deal with chores; the considered fairness notion is maximin share guarantee. In the divisible chores setting, Dehghani et al. (2018) give the first discrete and bounded protocol for envy-free chore division problem and Farhadi and Hajiaghayi (2018) prove the $\Omega(n \log n)$ lower bound for the number of queries in a proportional protocol for chores.

Finally, besides chores division and fair division with connectivity constraints, another stream of works that is related to our paper concerns approximation of fairness criteria. In the context of indivisible items, where the existence of fair allocations cannot be ensured, studying approximate fairness – like we do in this paper – is natural. Amongst the seminal works that concern this topic in the context of fair division of indivisible goods, Markakis and Psomas (2011) prove a worst case guarantee on the value that every agent can have and they propose a polynomial algorithm for computing allocations that achieve this guarantee. By contrast, they show that if $\mathsf{P} \neq \mathsf{NP}$ there is no polynomial algorithm to decide whether there exists an allocation where each agent can get a bundle worth at least $1/\rho n$ for any constant $\rho \geq 1$. Lipton et al. (2004) focus on the concept of envy-freeness. They show that there exists an allocation with maximum envy not exceeding the maximum marginal utility of a good. However, the problem of computing allocations with minimum possible envy is hard, even in the case of additive utilities.

Our contribution. In this paper, we extend the work of Bouveret et al. (2017) about fair division of goods on a graph. We also use the connectivity constraints defined by a graph on the items. However, we deal with *nondisposable undesirable* items, often called *chores*. We use three classic fairness criteria, namely proportionality, envy-freeness and equitability, and two different individual disutility aggregation functions: additive and maximum based. We show that dealing with goods and chores is inherently different. In particular, simply transforming an instance with chores into an instance with goods and then applying an algorithm that works for goods will not yield an allocation that satisfies the same properties in the initial chore instance.

Then we investigate the complexity of the problems to find a fair allocation of chores. It is known that these problems are hard on complete graphs in the additive case, but the maximum-based case, as far as we know, has not been studied before. Therefore, we complement the picture by providing efficient algorithms for proportionality and equitability, and show that envy-freeness leads to an NP-complete problem.

Further, we concentrate on two special classes of graphs: paths and stars. In more detail, we provide a general reduction for paths that directly implies NP-completeness of the existence problems for all the considered fairness criteria and both disutility aggregations. Moreover, by a very small modification of the reduction we obtain that these problems are hard even in the binary case *i.e.*, when disutility values for chores are either 0 or 1.

By contrast, if the underlying graph is a star, we propose an efficient algorithm, based on bipartite matching techniques, to decide whether an allocation exists such that each agent has a connected bundle whose disutility is 0. This in turn implies that envy-freeness and equitability criteria admit efficient algorithms for decision problems in the binary case. Matching techniques lead to efficient algorithms also in the maximum-based case, even when disutilities are not restricted to be binary. In the additive case we provide an efficient algorithm for proportionality. On the other hand, it is NP-complete to decide the existence of envy-free or equitable valid allocations on a star.

Outline. This paper is organized as follows. In Section 2 we introduce the model of connected fair division of indivisible chores and the definitions of the various fairness criteria used in the paper. Section 3 is devoted to a detailed comparison of the goods and chores setting. Our technical results are presented mainly in Sections 4, 5 and 6 which respectively deal with the cases where the underlying graph is a complete graph, a path and a star. Table 1 shows an overview of the results obtained in this paper. Finally, we discuss our findings and suggest some open problems in Section 7.

	complete graph		path		star	
	additive	maximum	additive	maximum	additive	maximum
proportionality	NPC	P	NPC	NPC[a]	P	P
	Proposition 3	Theorem 1	Theorem 4	Theorem 6	Theorem 8	Theorem 7
envy-freeness	NPC	NPC	NPC	NPC[a]	NPC[b]	NPC[b]
	Proposition 3	Theorem 3	Theorem 5	Theorem 6	Theorem 12	Theorem 13
equitability	NPC	P	NPC	NPC[a]	NPC	P
	Proposition 3	Theorem 2	Theorem 5	Theorem 6	Theorem 14	Theorem 9

[a] Even with binary disutilities
[b] Polynomial with strict disutilities, Theorems 10 and 11

Table 1: Overview of the complexities for the existence problems

2 Model

Let $N = \{1, 2, \dots, n\}$ be the set of agents, and let $G = (V, E)$ be an undirected graph. Vertices V represent objects, and they are interpreted as nondisposable chores. We will denote the number of chores by m. Each agent $i \in N$ has a non-negative disutility (cost, regret) function $u_i : V \to \mathbb{R}_+$. The n-uple of disutility functions is denoted by $\mathscr{U}$.

An instance of CONNECTED CHORE DIVISION CCD is a triple $\mathscr{I} = (N, G, \mathscr{U})$. When we shall occasionally talk about problems with positively interpreted utility, we shall call them CONNECTED FAIR DIVISION problems, briefly CFD.

Any subset $X \subseteq V$ is called a *bundle*. We consider two disutility aggregation functions. In the *additive* case the disutility agent i derives from bundle X is equal to the sum of the disutilities of the objects that form the bundle, *i.e.* $u_i^{add}(X) = \sum_{v \in X} u_i(v)$. In the *maximum-based* case the disutility of a bundle is derived from the maximum disutility of an object in the bundle, *i.e.* $u_i^{max}(X) = \max\{u_i(v) \mid v \in X\}$. If the aggregation function is not specified or if it is clear from the context, the superscript may be omitted. In the maximum-based extension we shall also consider an important *binary* case when the disutilities of agents for objects are either 1 or 0. The binary case represents the situation of agents finding some objects negative without expressing the “degree of negativity” and some other objects bring them neither nuisance nor joy.

In the additive case we assume that the disutilities are normalized. This means that there is a constant U such that $u_i^{add}(V) = U$ for each agent. Throughout the paper, unless stated otherwise, we will assume that $U = 1$.

An *allocation* is a function $\pi : N \to 2^V$ assigning each agent a bundle of objects. An allocation π is *valid* if:

- for each agent $i \in N$, bundle $\pi(i)$ is connected in G;
- π is complete *i.e.*, $\bigcup_{i\in N} \pi(i) = V$ and;
- no item is allocated twice, so that $\pi(i) \cap \pi(j) = \emptyset$ for each pair of distinct agents $i, j \in N$.

We say that a valid chore allocation π is:

- *proportional* if $u_i(\pi(i)) \leq \frac{U}{n}$ for all $i \in N$;
- *envy-free* if $u_i(\pi(i)) \leq u_i(\pi(j))$ for all $i, j \in N$;
- *equitable* if $u_i(\pi(i)) = u_j(\pi(j))$ for all $i, j \in N$.

Let us remind the reader that the corresponding notions of proportionality and envy-freeness for CFD are defined by reversing the respective inequalities. Equitability is defined in the same way in both cases, hence an allocation that is equitable for goods is equitable in the chores setting and conversely. However, no result for equitable allocation of goods with connected bundles has been published yet, so our results offer an analysis for the goods setting too.

Similarly as in the 'classic' case of indivisible goods without connectivity constraints, the existence of allocations fulfilling the above definitions (proportionality, envy-freeness and equitability) is not ensured in general. Therefore, we shall deal also with approximate fairness. For a given constant $\rho \geq 1$, we say that a valid chore allocation π is ρ-proportional if each agent $i \in N$ receives a bundle such that $u_i(\pi(i)) \leq \rho \cdot \frac{U}{n}$. An allocation π is ρ-envy-free if $u_i(\pi(i)) \leq \rho \cdot u_i(\pi(j))$ for each pair of agents i, j. Finally, an allocation is ρ-equitable if $u_j(\pi(j))/\rho \leq u_i(\pi(i)) \leq \rho \cdot u_j(\pi(j))$ holds for each pair of agents i, j. Later in this paper we shall see that even for paths the problems to decide whether a valid allocation exists such that the disutility of each agent equals 0 is NP-hard. This immediately implies that the problems to decide the existence of approximately fair allocations are intractable for any $\rho \geq 1$.

We will consider the following computational problems that all take an instance $\mathscr{I} = (G, N, \mathscr{U})$ of CCD as their input. PROP-CCD, EF-CCD and EQ-CCD ask whether $\mathscr{I}$ admits a proportional, envy-free and equitable allocation, respectively. If we want to stress which disutility aggregation functions is used, we insert prefix ADD or MAX to this notation.

Notice that with maximum-based disutility aggregation, proportionality does not have a similar interpretation as in the additive case, where dividing the disutility by the number of agents corresponds to sharing the total burden. Still, we shall use the term *proportionality* also in the case when agents care for the worst item in their bundle, meaning that we seek an allocation that restricts the disutility by the same threshold for everybody. We shall use the notation λ-MAX-PROP-CCD to denote the problem to decide whether for a given instance of CCD there exists a valid allocation π such that $u_i^{max}(\pi(i)) \leq \lambda$ for each agent $i \in N$.

It is easy to see that all the considered problems belong to the class NP, as given an allocation, it can be verified in polynomial time whether it is valid and also whether it is proportional, equitable (linear in the problem size) or envy-free.

Let us conclude these technical preliminaries with a brief recall of the definitions of the basic graph-theoretic notions we use in the paper. A graph (V', E') is a *subgraph* of (V, E) if $V \subseteq V'$ and $E \subseteq E'$. A sequence of vertices $(v_0, v_1, \ldots, v_k)$ of G such that $\{v_{i-1}, v_i\} \in E$ for each $i = 1, 2 \ldots, k$ is called a *path*. If additionally $\{v_k, v_0\} \in E$, it is called a *cycle*. A graph is said to be *connected* if there is a path between each pair of vertices. A *connected component* of a graph G is an inclusion wise maximal connected subgraph of G. A connected graph is a *tree* if it does not contain any cycle. A *star* with n vertices is a tree with $n-1$ vertices of degree one, and one vertex (the *center*) of degree $n-1$.

A graph (V,E) is *bipartite* if there is a partition V_1, V_2 of its vertices such that for all $\{v_1, v_2\} \in E$, $e_1 \in V_1$ and $e_2 \in V_2$. In the following, we will slightly abuse notations and denote any bipartite graph with a triple (V_1, V_2, E). A *matching* M of a bipartite graph $G = (V_1, V_2, E)$ is a subgraph of G such that no two edges in M share a common vertex. A matching is *perfect* if it contains exactly $\min(|V_1|, |V_2|)$ edges.

Finally, a *flow network* is a quadruple (V, A, lb, ub), where (V, A) is a directed graph where one vertex denoted by σ (the source) has no incoming arc, and one vertex denoted by τ (the sink) has no outgoing arc. Functions *ub* and *lb* map each arc (v, v') to an integer, such that $lb(v, v') \leq ub(v, v')$ for all $(v, v') \in A$. $ub(v, v')$ is an upper bound, called the *capacity* of (v, v') and $lb(v, v')$ the *lower bound* of (v, v'). If the lower bound is omitted for an arc then it means that it is 0 for this arc. For each vertex $v \in V$, let $\delta_+(v)$ ($\delta_-(v)$, respectivelly) denote the set of vertices which are connected to v in (V, A) through an outgoing edge from v (an ingoing edge to v, respectively). A *valid flow* of a flow network (V, A, lb, ub) is a mapping $f : A \to \mathbb{R}$ satisfying the following two conditions:

- for each $(v, v') \in A$, $lb(v, v') \leq f(v, v') \leq ub(v, v')$ (lower bound and capacity constraints);
- for each $v \in V \setminus \{\sigma, \tau\}$, $\sum_{v' \in \delta_-(v)} f(v', v) = \sum_{v' \in \delta_+(v)} f(v, v')$ (flow conservation).

For such a feasible flow, the value $\sum_{v' \in V} f(\sigma, v')$ is called the *value of the flow*. A maximum flow is a flow with maximum value of all feasible flows.

Section 6.7. of Ahuja et al. (1993) explains how to decide the existence of a feasible flow in a network with nonzero lower bounds by one computation of a maximum flow in a network with zero lower bounds. For the latter problem many efficient algorithms exist (see e.g. Chapter 10 of Schrijver 2003). The first one was proposed by Dinits (1970) and its complexity is $O(pq^2)$ where $p = |V|$ and $q = |A|$.

We shall also intensively use the fact that if a network with integral capacities and lower bounds admits a feasible flow of integral size K then it also admits an *integral* flow of size K – Integrality Lemma, see Corollary 11.2c or Theorem 11.1 in the book by Schrijver (2003).

3 Relation between fair division of goods and chores

Taking into account the large number of existing results concerning the fair division of goods, one could be tempted to try and adapt these results to the case of chores. For two agents, this will certainly work. As Bogomolnaia et al. (2017, page 4) explain, in this special case, allocating goods is equivalent to allocating exemptions of chores. More formally, the following approach will work. Pretend that the disutilities are utilities and apply any fair division algorithm. If the obtained allocation π is proportional then simply exchanging the bundles yields a chore-proportional division, since $u_i(\pi(i)) \geq 1/2$ implies $u_i(\pi(3-i)) \leq 1/2$. Similarly, envy-freeness for goods means $u_i(\pi(i)) \geq u_i(\pi(3-i))$, and by exchanging the bundles we get envy-freeness for chores, as $u_i(\pi(2-i)) \leq u_i(\pi(i))$.

However, as soon as there are three agents or more, this approach does not work, and there is no obvious equivalence between CCD instances and CFD instances, as we will now illustrate. A first easy observation is that if in a CFD instance there are more agents than items, no proportional and envy-free allocation can exist (as necessarily somebody receives nothing), which is not necessarily the case with chores. It may seem natural to transform a chore division instance to a 'dual' fair division instance by simply replacing each disutility $u_i(v)$ by a 'reverse' desirable utility $M - u_i(v)$ for each agent i and each object v, where M is a suitable number. We show that the properties of the mutually dual instances do not translate.

Example 1 Let us consider the CCD instance $\mathscr{I}$ with three agents 1,2,3 and four vertices v_1, v_2, v_3, v_4 arranged on a path in this order and disutilities given in the left half of Table 2. Its right half shows the utilities for the 'dual' CFD instance $\mathscr{I}'$. [2]

	v_1	v_2	v_3	v_4
agent 1	6	4	0	0
agent 2	7	0	1	2
agent 3	5	0	0	5

	v_1	v_2	v_3	v_4
agent 1	4	6	10	$10^\star$
agent 2	$3^\star$	$10^\star$	9	8
agent 3	5	10	$10^\star$	5

Table 2: The CCD (left) and CFD (right) instances for proportionality notion

As $U = 10$, a proportional chore allocation should give each agent a bundle of disutility at most 10/3. $\mathscr{I}$ does not admit a valid proportional allocation, as nobody is willing to take vertex v_1. In the dual CFD instance $\mathscr{I}'$, the proportional share is 10 and a proportional valid allocation exists: simply give agent 1 vertex v_4, agent 2 bundle $\{v_1, v_2\}$ and agent 3 vertex v_3. This allocation is indicated by stars in the right half of Table 2.

Example 2 Now slightly change the disutilities of agent 3; the new CCD instance and its dual CFD instance are given in Table 3.

	v_1	v_2	v_3	v_4
agent 1	6	4	$0^\star$	$0^\star$
agent 2	7	$0^\star$	1	2
agent 3	$0^\star$	5	5	0

	v_1	v_2	v_3	v_4
agent 1	4	6	10	10
agent 2	3	10	9	8
agent 3	10	5	5	10

Table 3: The CCD (left) and CFD (right) instances for envy-freeness.

Now $\mathscr{I}$ has an envy-free valid allocation, namely $\pi(1) = \{v_3, v_4\}$, $\pi(2) = \{v_2\}$ and $\pi(3) = \{v_1\}$, again indicated by stars in the left half of Table 3. However, there is no envy-free valid allocation in the dual fair division instance $\mathscr{I}'$. To see this, let us first realize that as there are three agents and four items, exactly one of the agents has to receive a bundle consisting of two vertices. Since each allocated bundle has to be connected, there are exactly three such two-elements bundles: $\{v_1, v_2\}$, $\{v_2, v_3\}$ and $\{v_3, v_4\}$. One can see that each such bundle has utility strictly greater than 10 for at least two agents, and as each agent values individual vertices at not more than 10, there will always be somebody envying the agent receiving the two-element bundle.

In the previous examples, we have mapped a CCD instance to a CFD one using a simple linear transformation of the disutilities to utilities. The next two propositions are much

[2] In this and the following example the disutilities in the CCD instances (tables in the left) are normalized to 10 and the utilities in the corresponding dual CFD instances to 30 (tables in the right).

stronger, we show that there is no transformation of the set of CCD instances to the set of CFD instances that preserves the fairness properties under additive aggregation. For any function φ, we will denote by $\varphi(\mathscr{U})$ the n-tuple of functions $(\varphi(u_1),\ldots,\varphi(u_n))$.

Proposition 1 *There is no mapping $\varphi : [0,1] \to \mathbb{R}_+$ such that a CCD instance $\mathscr{I} = (N,G,\mathscr{U})$ admits an envy-free allocation if and only if the CFD instance $\mathscr{J} = (N,G,\varphi(\mathscr{U}))$ admits an envy-free allocation. This assertion holds even if restricted to binary CCD instances.*

Proof For contradiction assume that such a mapping φ exists. We provide three different instances $\mathscr{I}_1,\mathscr{I}_2,\mathscr{I}_3$ of CCD, described in Figure 1, which will lead to a contradiction.

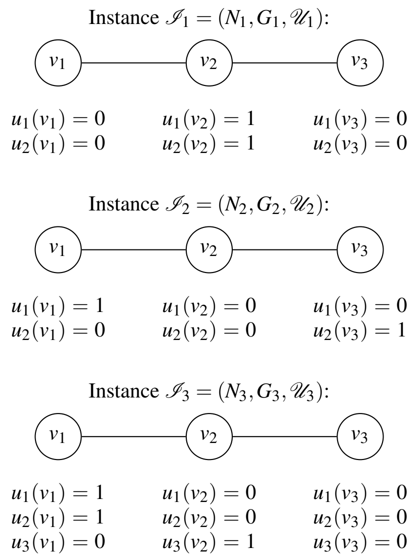

Fig. 1: The CCD instances used in the proof of Proposition 1.

In $\mathscr{I}_1$ we have $n = 2$ and $m = 3$ (Figure 1, top). It is clear that $\mathscr{I}_1$ admits no envy-free allocation, since the agent who receives v_2 necessarily envies the other agent. If φ is a desired mapping then the CFD instance $\mathscr{J}_1 = (N_1, G_1, \varphi(\mathscr{U}_1))$ admits no envy-free allocation. This means in the allocation π' defined by $\pi'(1) = \{v_1, v_2\}$ and $\pi'(2) = \{v_3\}$ either agent 1 envies agent 2 (this means $\varphi(0) + \varphi(1) < \varphi(0)$) or agent 2 envies agent 1 (this means $\varphi(0) < \varphi(1) + \varphi(0)$). As the values of φ are nonnegative, the former is not possible and that latter leads to $\varphi(1) > 0$.

The second instance $\mathscr{I}_2$ also has $n = 2$ and $m = 3$ (Figure 1, middle). $\mathscr{I}_2$ admits an envy-free allocation $\pi(1) = \{v_2, v_3\}$, $\pi(2) = \{v_1\}$, hence the corresponding CFD instance $\mathscr{J}_2 = (N_2, G_2, \varphi(\mathscr{U}_2))$ should also admit an envy-free allocation; let us denote one by π'. As we have shown $\varphi(1) > 0$ previously, π' cannot allocate V to a single agent, since the other one will envy her. Further, for the same reason, $v_1 \in \pi'(2)$ is impossible. Namely, if $\pi'(2) = \{v_1\}$ then agent 2 envies 1 and if $\pi'(2) = \{v_1, v_2\}$ then agent 1 envies 2. Hence we must have $v_1 \in \pi'(1)$, $v_3 \in \pi'(2)$ and object v_2 is either allocated to agent 1 or 2. In both cases, for π' to be envy-free, the following inequalities both have to be fulfilled: $\varphi(1) + \varphi(0) \geq \varphi(0)$ (trivial) and $\varphi(1) \geq 2 \cdot \varphi(0)$.

The third instance $\mathscr{I}_3$ has $n = 3$ and $m = 3$ (Figure 1, bottom). $\mathscr{I}_3$ admits an envy-free allocation π with $\pi(1) = \{v_3\}$, $\pi(2) = \{v_2\}$ and $\pi(3) = \{v_1\}$. Let us consider the corresponding CFD instance $\mathscr{J}_3 = (N_3, G_3, \varphi(\mathscr{U}_3))$. Since $\varphi(1) > 0$, in any envy-free allocation π' of $\mathscr{I}_3$ each agent must receive a nonempty bundle, hence each one gets a single item. Further object v_1 can be given to only one agent, say agent i. Neither agent 1 nor agent 2 should envy agent i, therefore we must have $\varphi(0) \geq \varphi(1)$. However, this is in contradiction with inequalities $\varphi(1) \geq 2 \cdot \varphi(0)$ and $\varphi(1) > 0$ derived previously. □

We have a similar result for proportionality:

Proposition 2 *There is no mapping $\varphi : [0,1] \to \mathbb{R}_+$ such that a CCD instance $\mathscr{I} = (N, G, \mathscr{U})$ admits a proportional allocation if and only if the CFD instance $\mathscr{J} = (N, G, \varphi(\mathscr{U}))$ admits a proportional allocation.*

Proof For contradiction assume that such a function φ exists. We provide in Figure 2 two different instances of CCD which will lead to a contradiction.

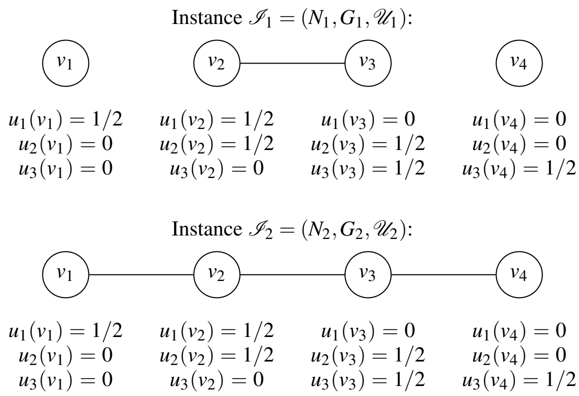

Fig. 2: The CCD instances used in the proof of Proposition 2.

In $\mathscr{I}_1$ we have $n = 3$ and $m = 4$ (Figure 2, top). $\mathscr{I}_1$ does not contain any proportional allocation since in any valid allocation one agent should receive bundle $\{v_2, v_3\}$ and each agent has a disutility at least $1/2$ for this bundle (greater than the proportional share $1/3$).

Let us now take the corresponding CFD instance $\mathscr{J}_1 = (N_1, G_1, \varphi(\mathscr{U}_1))$. No valid allocation in $\mathscr{J}_1$ should be proportional. As the utility of V for each agent is equal to $U = 2(\varphi(0) + \varphi(1/2))$, this means that in each valid allocation at least one agent has utility smaller than $\frac{2}{3} \cdot (\varphi(0) + \varphi(1/2)$. Consider the allocation $\pi'(1) = \{v_2, v_3\}$, $\pi'(2) = \{v_4\}$ and $\pi'(3) = \{v_1\}$. If π' is not proportional then either $\varphi(0) + \varphi(1/2) < \frac{2}{3} \cdot (\varphi(0) + \varphi(1/2))$ or $\varphi(0) < \frac{2}{3} \cdot (\varphi(0) + \varphi(1/2))$. As the values of function φ are nonegative, the former is not possible, so we have

$$\varphi(0) < \frac{2}{3} \cdot (\varphi(0) + \varphi(1/2)) \text{ or, equivalently, } \varphi(0) < 2 \cdot \varphi(1/2). \tag{1}$$

Further, neither should allocation $\pi''(1) = \{v_1\}$, $\pi''(2) = \{v_2, v_3\}$ and $\pi''(3) = \{v_4\}$ be proportional. The utility of at least one agent should be too small, so we have that either

$\varphi(1/2) < \frac{2}{3}\cdot(\varphi(0)+\varphi(1/2))$ or $2\cdot\varphi(1/2) < \frac{2}{3}\cdot(\varphi(0)+\varphi(1/2))$. The second possibility is equivalent to $\varphi(0) > 2\cdot\varphi(1/2)$, which is by (1) impossible. Therefore we have

$$\varphi(1/2) < \frac{2}{3}\cdot(\varphi(0)+\varphi(1/2)) \text{ or, equivalently, } \varphi(1/2) < 2\cdot\varphi(0). \tag{2}$$

In the second instance $\mathscr{I}_2$ we also have $n = 3$ and $m = 4$ (Figure 2, bottom). $\mathscr{I}_2$ contains a proportional allocation π with $\pi(1) = \{v_3, v_4\}$, $\pi(2) = \emptyset$ and $\pi(3) = \{v_1, v_2\}$. For the corresponding CFD instance $\mathscr{J}_2 = (N_2, G_2, \varphi(\mathscr{U}_2))$ notice that in any valid allocation at least one agent receives no more than one object, and hence her utility is either $\varphi(0)$ or $\varphi(1/2)$. Inequalities (1) and (2) thus imply that $\mathscr{J}_2$ does not admit any valid proportional allocation, leading to a contradiction. □

The examples and assertions in this section show that in general it is impossible to derive algorithms for finding fair chore allocations from their counterparts for fair allocations of positive items, and so one can naturally expect that the complexity results of the corresponding problems in the CFD and CCD setting might differ. This motivates our subsequent complexity analysis.

4 Complete graphs

Note that the classic case (without connectivity requirements) corresponds in our model to the underlying graph being complete. Then, all the problems studied are hard for the additive disutility aggregation. The intractability can be proved using a reduction from PARTITION (see *e.g.* Demko and Hill 1998).

Proposition 3 ADD-PROP-CCD*,* ADD-EF-CCD *and* ADD-EQ-CCD *are* NP*-complete even for two agents with the same disutility function.*

Interestingly, if we use the maximum operator instead of the sum to aggregate the disutilities, the complexity landscape changes. More precisely, suppose now that agents aggregate their disutilities using maximum and consider the *greedy* algorithm that consists in allocating each chore v to any agent in $\arg\min_{j\in N} u_j(v)$. Then we claim that this algorithm computes an allocation $\hat{\pi}$ that minimizes $\max_{i\in N} u_i^{max}(\pi(i))$ over all allocations. This can be used to prove the following result:

Theorem 1 λ-MAX-PROP-CCD *can be solved in polynomial time for any* λ*.*

Proof Let us first prove that the greedy algorithm described above computes an allocation that minimizes $\max_{i\in N} u_i^{max}(\pi(i))$.

Let $\hat{\pi}$ be the obtained allocation. Let $v^* = \arg\max_{v\in V}\min_{i\in N} u_i(v)$ and let i^* be the agent who receives v^* in $\hat{\pi}$. For each chore v, the disutility incurred for v by the agent who receives it in $\hat{\pi}$ is $\min_{j\in N} u_j(v)$. Hence, the disutility of each agent in $\hat{\pi}$ is bounded above by $\max_{v\in V}\min_{i\in N} u_i(v) = u_{i^*}(v^*)$. Let π be another allocation, and let i be the agent who receives v^* in π. By definition, $u_i(v^*) \geq u_{i^*}(v^*)$, and hence, the highest disutility in π is not smaller than $u_{i^*}(v^*)$, which is the highest disutility in $\hat{\pi}$, which proves that $\hat{\pi}$ is optimal with respect to the highest disutility of an agent.

Hence an instance of λ-MAX-PROP-CCD is a Yes-instance if and only if the greedy algorithm computes an allocation where the greatest disutility of an agent is at most λ. □

Theorem 2 MAX-EQ-CCD *can be solved in polynomial time.*

Proof Let $\mathscr{I} = (G, N, \mathscr{U})$ be a CFD instance. We first check by greedy algorithm whether there exist an allocation giving each agent a bundle (possibly empty) with disutility 0. If this is not the case, we use Algorithm 1.

Algorithm 1: Computing an equitable allocation for max valuations

Input: $\mathscr{I} = (G, N, \mathscr{U})$
Output: an equitable allocation π or $\emptyset$

1 $\pi \leftarrow (\emptyset, \dots, \emptyset)$;
2 **foreach** $\eta \in \{u_i(v) \mid i \in N, v \in V\} \setminus \{0\}$ **do**
3     $H_\eta \leftarrow (N, V, L_\eta)$, a bipartite graph with $\{i, v\} \in L_\eta$ if and only if $u_i(v) = \eta$;
4   **if** *exists a matching M covering N* **then**
5   **foreach** $i \in N$ **do**
6             $\pi(i) \leftarrow \{v \mid \{i, v\} \in M\}$;
7   **if** $\forall v \in V$ *not covered by M exists $i \in N$ such that $u_i(v) \leq \eta$* **then**
8   **foreach** $v \in V \setminus \{v' \in V \mid \exists i \in N, \{i, v'\} \in M\}$ **do**
9   **choose** *any i such that $u_i(v) \leq \eta$*;
10                 $\pi(i) \leftarrow \pi(i) \cup \{v\}$;
11   **return** π
12 **return** $\emptyset$

Now we claim that Algorithm 1 returns an equitable allocation with a positive common value of the disutility if and only one there exists one. Let the obtained allocation be π. Then, obviously, π is complete, since each vertex v is allocated, either on Line 6 or on Line 10. Moreover, the share of each agent i only contains chores of disutility either equal η (chores allocated on Line 6) or not exceeding η (chores allocated on Line 10). Hence, $u_i(\pi(i)) \leq \eta$. Finally, since the matching M computed on Line 4 covers N, each agent receives at least one chore of disutility η. Therefore, $u_i(\pi(i)) = \eta$ for all $i \in N$, which proves that π is equitable.

Conversely, suppose that there is an equitable allocation π that gives disutility η to each agent. Then, by definition of the maximum based disutility, either $\eta = 0$ and we obtained a desired allocation by the greedy algorithm or $\eta \in S = \{u_i(v) \text{ for } i \in N \text{ and } v \in V\} \setminus \{0\}$. Moreover, for each $i \in N$, at least one chore $v_i \in \pi(i)$ has disutility η. Since π is a valid allocation, all the v_i are distinct. Thus, $M = \{\{i, v_i\}, i \in N\}$ is a matching of H_η that covers N. All other vertices are allocated to agents valuing them not more than η (otherwise π would not be equitable), which proves that condition of Line 7 is satisfied, and that Algorithm 1 returns allocation π, which is equitable.

The size of set S (hence the number of iterations in the global loop) is upper-bounded by $n \times m$. For each iteration, graph H_η has $m + n$ vertices and at most $n \times m$ edges. A maximum cardinality matching in a bipartite graph with p vertices and q edges can found by Hopcroft-Karp algorithm (Schrijver 2003) in time $O(\sqrt{p}\, q)$, which in our case gives $O(mn\sqrt{m+n})$ and the rest of the loop just runs through the chore-agent pairs, hence its complexity is bounded by $O(nm)$. The total complexity of Algorithm 1 is therefore $O(n^2 m^2 \sqrt{m+n})$, which is polynomial in the input size. □

By contrast, envy-freeness criterion leads to an intractable problem:

Theorem 3 MAX-EF-CCD *is NP-complete even if the underlying graph G is complete.*

Before giving the formal proof, we introduce the (2,2)-E3-SAT problem, that will be used in the next proof, as well as in the main construction in Section 5 and in the proofs of

Theorem 12. This problem, which has been proved to be NP-complete (Berman et al. 2003), is defined as follows.

Instance $\mathscr{I}$**:** A Boolean formula F in Conjunctive Normal Form, such that each clause in F has size three, and each variable occurs exactly twice unnegated and exactly twice negated.
Question: Is F satisfiable?

We shall assume that the given formula F as an instance of (2,2)-E3-SAT consists of clauses $C = \{c_1, \ldots, c_t\}$ containing a set of variables $X = \{x_1, \ldots, x_s\}$. We will denote by L the set of literals in F, i.e., $L = \bigcup_{j=1}^{s} \{x_j^1, x_j^2, \bar{x}_j^1, \bar{x}_j^2\}$, where x_j^1 (resp. x_j^2) denotes the first (resp. second) positive occurrence of variable x_j and $\bar{x}_j^1$ (resp. $\bar{x}_j^2$) denotes the first (resp. second) negative occurrence of the same variable. Further, L_i will denote the set of literals in clause c_i and for any literal $\ell \in L$ we denote by $c(\ell)$ the clause containing literal ℓ. Notice that the structure of the formula implies $3t = 4s$ and hence $s = 3t/4$.

Proof (Theorem 3) To prove NP-completeness, we provide a polynomial reduction from (2,2)-E3-SAT.

For each an instance of (2,2)-E3-SAT, i.e., a formula F that has the structure as described above, we construct an instance of MAX-EF-CCD as follows. The set of chores is $V = W \cup Z \cup Z' \cup Y$, where $W = \bigcup_{j=1}^{s} W_j$ with $W_j = \{w_j^1, w_j^2, \bar{w}_j^1, \bar{w}_j^2\}$ are literal chores, $Z = \{z_1, z_2, \ldots, z_s\}$ are clause chores, $Z' = \{z'_1, z'_2, \ldots, z'_s\}$ are dummy clause chores and $Y = \bigcup_{j=1}^{s} Y_j$, with $Y_j = \{y_j^1, y_j^2, y_j^3, y_j^4\}$ are dummy variable chores. The literal chore corresponding to literal $\ell \in L$ will be denoted $w(\ell)$.

We further assume that the set of chores is ordered $W_1, W_2, \ldots, W_s, Z, Z', Y_1, Y_2, \ldots, Y_s$ while the ordering within each subset is the same as the order in which the chores in the respective subset have been written above. Let $\beta(v)$ denote the position of a chore $v \in V$ in this ordering.

The set of agents is $N = B \cup B' \cup P \cup Q$, where $B = \{b_1, b_2, \ldots, b_t\}$ are clause agents, $B' = \{b'_1, b'_2, \ldots, b'_t\}$ are dummy clause agents, $P = \{p_1, p_2, \ldots, p_s\}$ are variable agents and $Q = \bigcup_{j=1}^{s} Q_j$ with $Q_j = \{q_j^1, q_j^2, q_j^3, q_j^4\}$ are dummy variable agents.

The disutilities are defined in Table 4, where for each agent we list the chores with disutilities equal to 0 and to ε, where $0 < \varepsilon < 1$ is fixed. The disutility of any chore v to agent who does not have v displayed in this table is equal to $\beta(v)$.

agent	chores with disutility equal 0	chores with disutility equal ε
$b_i, i = 1,2,\ldots,t$	$z_i, \{w(\ell) \mid \ell \in L_i\}$	–
$b'_i, i = 1,2,\ldots,t$	z_i	z'_i
$p_j, j = 1,2,\ldots,s$	$w_j^1, w_j^2, \bar{w}_j^1, \bar{w}_j^2$	–
$q_j^1, j = 1,2,\ldots,s$	$w_j^1, \bar{w}_j^1$	y_j^1
$q_j^2, j = 1,2,\ldots,s$	$w_j^1, \bar{w}_j^2$	y_j^2
$q_j^3, j = 1,2,\ldots,s$	$w_j^2, \bar{w}_j^1$	y_j^3
$q_j^4, j = 1,2,\ldots,s$	$w_j^2, \bar{w}_j^2$	y_j^4

Table 4: Disutilities in the proof of Theorem 3.

Let us briefly explain how the reduction works before proving that it is correct. Each variable agent will receive a subset of literal chores that correspond to her variable. Each dummy variable agent will receive her corresponding dummy variable chore and will envy the corresponding variable agent as soon as she does not receive a subset of literal chores containing either two positive literals or two negative literals. Each clause agent will receive a clause chore as well as at least one literal chore associated with one of her literals. Clause chores ensure that no variable agent will envy a clause agent. Furthermore, each dummy clause agent will receive her corresponding dummy clause chore and will envy her corresponding clause agent as soon as she does not receive at least one of her corresponding literal chores.

Assume first that f is a truth assignment that satisfies all clauses in C. We construct from f an assignment π as follows. For each variable x_j, if x_j is `false` according to f then $\pi(p_j) = \{w_j^1, w_j^2\}$, otherwise $\pi(p_j) = \{\bar{w}_j^1, \bar{w}_j^2\}$. Furthermore, $\pi(b_i) = \{z_i\} \cup \{w(\ell), \ell \in L_i$ and ℓ is `true` in $f\}$. Finally, $\pi(b'_i) = \{z'_i\}$ for $i = 1,2,\dots,t$ and $\pi(q_j^k) = \{y_j^k\}$ for $j = 1,2,\dots,s$ and $k = 1,2,3,4$.

Let us first see that π is valid. Clearly, no chore is assigned to more than one agent. Moreover, no chore remains unassigned, as each chore $y_j^k \in Y$ is assigned to agent $y_j^k \in Q$, each $z'_j \in Z'$ to b'_j, each $z_j \in Z$ to b_j and each chore in W that corresponds to a literal ℓ that is `true` according to f is assigned to $b_{c(\ell)}$ and if it corresponds to a `false` literal of variable x_j then it is is assigned to agent p_j.

Now we argue that π is envy free. Namely, the agents in $B \cup P$ receive in π bundles with disutility 0, so they do not envy. Dummy agents receive bundles with disutility ε. Take a dummy clause agent $b' \in B$. She could only envy an agent that receives chore z_i, but as this chore is assigned to agent b_i together with at least one chore in W (that corresponds to a `true` literal in clause c_i), there is no envy. Dummy variable agent $q_j^\ell \in Q$ could envy agents $p_j \in P$, but the bundle $\pi(p_j)$, which is either $\{w_j^1, w_j^2\}$ or $\{\bar{w}_j^1, \bar{w}_j^2\}$, contains at least one chore v such that $u_{q_i}(v)$ is a strictly positive integer, hence greater than ε, so there is no envy here too.

Conversely, suppose that there is a valid assignment π such that no agent envies another one. In the first part of the proof we use mathematical induction on the reverse ordering $\beta(v)$ of the chores in the following way: we take the next chore v and argue that v must belong to the bundle of a certain agent i. Let us say that agent i was *treated*.

As the disutility of y_s^4 is maximum of all chores, if $y_s^4 \in \pi(a)$ for any agent $a \neq q_s^4$ then a will envy any other agent. Therefore $y_s^4 \in \pi(q_s^4)$. Now suppose that $y_j^k \in \pi(q_j^k)$ for each $y_j^k \in Y$ such that $\beta(y_j^k) > u$. Take y_j^k such that $\beta(y_j^k) = u$. If $y_j^k \in \pi(a)$ for some agent $a \neq q_j^k$ then a will envy any other agent that was not treated yet.

Similarly, by induction for $i = t, t-1, \dots, 1$ we show that $z'_i \in \pi(b'_i)$. As the disutility of z'_t is maximum of all chores that have not yet been assigned, if $z'_t \in \pi(a)$ for some agent $a \neq b'_t$ then a will envy any other agent in $N \setminus Y$. Therefore $z'_i \in \pi(b'_i)$. Now suppose that $z'_i \in \pi(b'_i)$ for each $i > k$. Take z'_k. If $z'_k \in \pi(a)$ for any agent $a \neq b'_k$ then a will envy any agent not treated so far.

By an analogical inductive argument we show that $z_i \in \pi(b'_i)$ or $z_i \in \pi(b_i)$ for each $i = t, t-1, \dots, 1$ because otherwise the agent that gets this chore will envy any untreated agent (for example, an agent in P).

Now we know that the disutility of each agent q_j^k in π is at least ε and q_j^k does not envy agent p_j. So we must have that either $\{w_j^1, w_j^2\} \subseteq \pi(p_j)$ or $\{\bar{w}_j^1, \bar{w}_j^2\} \subseteq \pi(p_j)$ for each $j = 1,2,\dots,s$. Let us say that x_j is `false` in the former case and that x_j is `true` in the latter case. Finaly, so as no agent b'_i envies b_i, we get that $\pi(b_i)$ must contain at least one chore

$w(\ell)$ for $\ell \in L_i$, and due to the truth values and assignment of chores in W defined above, this chore must correspond to a `true` literal in clause c_i. Hence we obtain an assignment of truth values that makes F `true`. □

5 Paths

Even if the underlying graph is restricted to be a path then all the considered chore division problems are intractable, as we now show. All the proofs in this section are based on the same construction starting from an instance F of (2,2)-E3-SAT.

For any fixed integer $\rho \geq 1$, we will construct an instance $\mathscr{I}_\rho$ of CCD with the set of chores $V = Y \cup Z \cup D$ where $Y = \{y_1, \ldots, y_t\}$ are *clause* chores, $Z = \cup_{j=1}^{s} \{z_j^1, z_j^2, \bar{z}_j^1, \bar{z}_j^2\}$ are *literal* chores and $D = \{d_1, \ldots, d_s\}$ are *variable* chores. The number of chores is thus $m = 5s + t$. The graph G defining the neighborhood relation between chores (illustrated in Figure 3) has edges:

- (y_i, y_{i+1}) for $i = 1, 2, \ldots, t-1$;
- (y_t, z_1^1);
- $(z_j^1, z_j^2); (z_j^2, d_j), (d_j, \bar{z}_j^1), (\bar{z}_j^1, \bar{z}_j^2)$ for $j = 1, 2, \ldots, s$;
- $(\bar{z}_j^2, z_{j+1}^1)$ for $j = 1, 2, \ldots, s-1$.

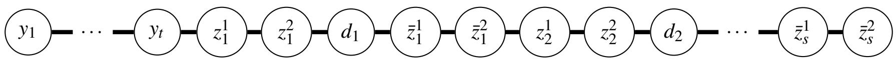

Fig. 3: The graph for Lemma 1

The set of agents in $\mathscr{I}_\rho$ is $N = B \cup P \cup Q \cup R$, where $B = \cup_{j=1}^{s} \{b_j^1, b_j^2, \bar{b}_j^1, \bar{b}_j^2\}$ are *literal* agents, $P = \{p_1, \ldots, p_s\}$, $Q = \{q_1, \ldots, q_s\}$ are *variable* agents and $R = \{r_1, \ldots, r_{\rho(5s+t)-6s+1}\}$ are dummy agents (observe that $\rho(5s+t) - 6s + 1 > 0$ for any $\rho \geq 1$ since $3t = 4s$ implies $t > s$). So the total number of agents in $\mathscr{I}$ is $n = \rho(5s+t) + 1$.

Let us remark that each literal $\ell \in L$ from formula F has in $\mathscr{I}_\rho$ its 'corresponding' chore in Z and agent in B; they will be denoted by $z(\ell)$ and $b(\ell)$, respectively.

For each agent $a \in N$ her disutility is 0 for some specific chores and the total disutility of 1 for agent a is distributed uniformly among the remaining chores, to achieve normalization. The details are given in Table 5.

agent	chores with disutility equal 0	disutility of the remaining chores
$b(\ell), \ell \in L$	$z(\ell), y_{c(\ell)}$	$1/(5s+t-2)$
$p_j, j = 1, 2, \ldots, s$	$z_j^1, z_j^2, \bar{z}_j^1, \bar{z}_j^2$	$1/(5s+t-4)$
$q_j = 1, 2, \ldots, s$	d_j	$1/(5s+t-1)$
$r_j, j = 1, 2 \ldots, \rho(5s+t) - 6s + 1$	–	$1/(5s+t)$

Table 5: Disutilities for the construction in Section 5.

Lemma 1 *If formula F is satisfiable then $\mathscr{I}_\rho$ admits a valid allocation π such that $u_a(\pi(a)) = 0$ for each agent $a \in N$. If F is not satisfiable then for any valid allocation π there exists an agent whose bundle has disutility greater than ρ/n.*

Proof Assume first that f is a truth assignment of F that satisfies all clauses in C. We construct from f a valid assignment of chores in $\mathscr{I}_\rho$ as follows. Assign each d_j to q_j. For each variable x_j assign agent p_j the bundle $\{z_j^1, z_j^2\}$ (the two chores are adjacent, so the bundle is connected) and the agents $\bar{b}_j^1, \bar{b}_j^2$ objects $\bar{z}_j^1$ and $\bar{z}_j^2$, respectively if x_j is `true` and assign agent p_j the bundle $\{\bar{z}_j^1, \bar{z}_j^2\}$ (again, this bundle is connected) and the agents b_j^1, b_j^2 chores z_j^1 and z_j^2, respectively if x_j is `false`. Finally, choose the first `true` literal $\ell \in L_i$ in each clause $c_i \in C$ and assign to the corresponding literal agent $b(\ell)$ chore y_i. Each agent in R is assigned an empty bundle, and also some agents $b(\ell)$ that correspond to `true` literals may receive an empty bundle. By checking Table 5, it is easy to see that each agent receives either nothing or a bundle whose disutility is 0, everybody receives a connected piece (if any) and that all chores are assigned.

Conversely, suppose that there is a valid assignment π of chores in $\mathscr{I}_\rho$ such that everybody receives a bundle with disutility at most ρ/n. Note first that $n = \rho(5s+t)+1$ implies $\rho = \frac{n-1}{5s+t} < \frac{n}{5s+t}$, which in turn implies $\frac{\rho}{n} < \frac{1}{5s+1}$. As any nonzero disutility is at least $1/(5s+t) > \rho/n$ this implies that each agent's bundle is either empty or has disutility 0. Therefore, chore d_j must be assigned to agent q_j. Further, for each i, chore y_i must be assigned to some agent $b(\ell) \in B$ that corresponds to a literal $\ell \in L_i$ contained in clause c_i. We now show that no two literal agents, corresponding to a literal and its negation, can both receive a clause chore of Y. For contradiction, suppose that for some j, agents b_j^k as well as $\bar{b}_j^{k'}$ (see the first row of Table 5), for $k, k' \in \{1,2\}$ are assigned some clause chores in Y. This means that chores z_j^k as well as $\bar{z}_j^{k'}$ must both be assigned to agent p_j since she is the only agent, in addition to b_j^k and $\bar{b}_j^{k'}$, who have a disutility 0 for these chores, and agent b_j^k (resp. agent $\bar{b}_j^{k'}$) cannot take z_j^k (resp. $\bar{z}_j^{k'}$) together with a chore of Y without either violating connectivity constraints or receiving a bundle of disutility greater than 0 (we can assume without loss of generality that literal x_1^1 is not part of clause c_t). But as chore d_j is assigned to agent q_j in any valid assignment where everybody has disutility 0, agent p_j gets a disconnected piece, which is a contradiction.

We now construct a truth assignment f for F as follows. For each clause c_j, the literal agent who receives chore y_j will define the truth value of its corresponding variable: if it is b_i^k for some $i \in \{1,\ldots,s\}$ and $k \in \{1,2\}$ then variable x_i is set to `true`, and if it is $\bar{b}_i^k$ for some $i \in \{1,\ldots,s\}$ and $k \in \{1,2\}$ then variable x_i is set to `false`. Because no two literal agents, corresponding to a literal and its negation, can both receive a chore of Y, a variable is not both set to `true` and `false`. Furthermore, this truth assignment satisfies all clauses of C. Finally, the truth value of the variables which have not been considered are set arbitrarily to complete the truth assignment. □

Lemma 1 directly implies the following result:

Theorem 4 *For any $\rho \geq 1$ the problem of deciding whether a given instance of* CCD *admits a ρ-proportional valid allocation is NP-complete, even if G is a path. In particular,* ADD-PROP-CCD *is NP-complete.*

As $m < n$ in $\mathscr{I}_\rho$, at least one agent gets nothing. Hence for each $\rho \geq 1$ in any ρ-envy-free or ρ-equitable allocation everybody has to get a bundle with disutility 0. So Lemma 1 directly implies also the following assertions.

Theorem 5 *For any $\rho \geq 1$ the problems of deciding whether a given instance of* CCD *admits a ρ-envy-free or ρ-equitable valid allocation are NP-complete, even G is a path. In particular, problems* ADD-EF-CCD *and* ADD-EQ-CCD *are NP-complete.*

For the maximum-based disutility extension, let us change the construction in the beginning of this section slightly. Namely, each positive disutility of an item will be set to 1. The same arguments as above are still valid, so we get the following assertion.

Theorem 6 MAX-EF-CCD, MAX-EQ-CCD *and* λ-MAX-PROP-CCD *for any $\lambda \in [0,1)$ are NP-complete if G is a path, even in the binary case.*

6 Stars

Let ω denote the center of the star. As each agent has to get a connected bundle, only the agent that is assigned ω can get more than one chore. According to the fairness criterion used, there are necessary conditions that each agent has to fulfill, so as to be entitled to be assigned ω; we shall call such an agent *central*. If is no agent fulfills these conditions then there is no valid allocation with the desired properties. If there exists such an agent, we still have to decide about the assignment of the leaves to the other agents. For this graph topology we first present efficient algorithms and then proceed to hard cases.

6.1 Easy cases

All the easy cases use a similar idea, borrowed from Bouveret et al. (2017). For each agent we check whether she can be the central agent. The central agent gets as many leaves as possible, and the assignment of the other leaves to other agents is found by using an efficient matching algorithm on bipartite graphs. Hence, all the problems studied in this section will have a similar complexity, which is determined by the matching algorithm used. The bipartite graph constructed by the algorithm has $O(m+n)$ vertices and $O(mn)$ edges. The Hopcroft-Karp matching algorithm applied to a graph with p vertices and q edges runs in $O(q\sqrt{p})$ steps, which in our case means $O(mn\sqrt{m+n})$ steps. Moreover, we might need to repeat the procedure for each agent, and that leads an overall complexity of $O(mn^2\sqrt{m+n})$.

Theorem 7 ADD-PROP-CCD *is solvable in polynomial time if G is a star.*

Proof First, an agent i can be central in a proportional allocation only if $u_i(\omega) \leq 1/n$. Let us check for each such agent i whether there is indeed a proportional valid allocation π assigning ω to i.

To this end, we create a bipartite graph $H = (Z, Z', L)$ with $Z = N \setminus \{i\}$, $Z' = V \setminus \{\omega\}$ and $\{j, v\} \in L$ if and only if $u_j(v) \leq 1/n$; the weight of this edge is $u_i(v)$.

We find a maximum weight matching M in H. A proportional valid allocation with i as a central agent exists if and only if the weight of M is at least $(n-1)/n$. If this is the case, assign the objects to agents in Z according to M and all the unmatched leaves plus the central vertex ω to i. □

A similar approach can be used to determine whether there is an allocation giving all agents a bundle where each item has a disutility at most λ. The following result for $\lambda = 0$ will be used in some of the subsequent algorithms.

Theorem 8 *If G is a star then the problem λ-MAX-PROP-CCD is polynomial for any λ.*

Proof To be able to decide the existence of such an allocation, let us first observe that an agent i can be central only if $u_i(\omega) \le \lambda$. For such an agent i let $X_i = \{v \in V; u_i(v) \le \lambda\}$. Now we create the bipartite graph $H = (Z, Z', L)$ where $Z = N \setminus \{i\}$, $Z' = V \setminus X_i$ and $\{j, v\} \in L$ if $u_j(v) \le \lambda$. Clearly, an allocation proving that the given instance is a Yes-instance of λ-MAX-PROP-CCD where i is the central agent exists if and only if H admits a matching M that covers all vertices in Z'. □

Theorem 9 MAX-EQ-CCD *is solvable in polynomial time if G is a star.*

Proof Using Theorem 8, we can decide in polynomial time whether there is a valid allocation such that everybody gets disutility 0. If this is the case, then we are done. If it is not, then we will run through all possible values η and determine, for each of them, whether there is a valid allocation such that everybody gets disutility η. As the individual disutility aggregator is the maximum, the only possible values for η are those among the set $\{u_i(v) \mid i \in N, v \in V\}$. Hence there are at most $m \times n$ possibilities for η.

An agent i can be a central agent in an equitable allocation with all disutilities equal to η only if $u_i(\omega) \le \eta$. We will handle the two cases $u_i(\omega) = \eta$ and $u_i(\omega) < \eta$ separately.

(i) $u_i(\omega) = \eta$. We create a flow network $H = (Z, L)$ as follows. Its vertices are the source σ, sink τ and one vertex for each agent and one vertex for each leaf of G. The source is connected to each agent vertex; capacities of arcs (σ, j) for $j \ne i$ are 1, capacity of arc (σ, i) is $m - n$. The choice of these capacities follows from the fact that each noncentral agent has to be assigned exactly one vertex and the central agent the remaning $m - n$ vertices.

There is an arc of capacity 1 between the vertex corresponding to agent $j \ne i$ and the vertex corresponding to leaf v if and only if $u_j(v) = \eta$ and between i and the vertex corresponding to leaf v if and only if $u_i(v) \le \eta$. Each vertex corresponding to a leaf is connected to τ, the capacities of these arcs are also 1. An allocation where each agent has disutility η for her bundle exists if and only if there is a flow of size $m - 1$ in this network, namely, the leaves are allocated to agents according to the agent-leaf arcs with nonzero flow.

(ii) $u_i(\omega) < \eta$. we have moreover to ensure that agent i gets at least one leaf v such that $u_i(v) = \eta$. The above construction of the flow network will be modified in the following way. Agent i will not be connected with leaves directly, but there will be two more vertices r_1, r_2 and the following arcs: (i, r_1) with capacity m and arcs (r_1, v) for each leaf v such that $u_i(v) = \eta$. Further, there is arc (i, r_2) with capacity $m - n - 1$ and arcs (r_2, v) for each leaf v such that $u_i(v) < \eta$. The construction of the flow network is shown in Figure 4. The choice of the capacities of the arcs outgoing from vertex i ensures that the arc (i, r_1) and consequently also arc (r_1, v) for some v with $u_i(v) = \eta$ will carry a nonzero flow, hence agent i will be assigned at least one leaf for which she has disutility equal η.

Again, an allocation where each agent receives a bundle of disutility η and where i is the central agent exists if and only if there is a flow of size $m - 1$ in this network. □

As we shall see later, the problem of deciding the existence of an envy-free valid allocation in the additive case is in general NP-complete. However, there is a plausible efficiently solvable special case. We say that *agents' preferences are strict on chores* if $u_i(v) \ne u_i(w)$ for any agent i and any pair of distinct chores v, w.

Theorem 10 ADD-EF-CCD *is solvable in polynomial time if G is a star and the agents' preferences are strict on chores.*

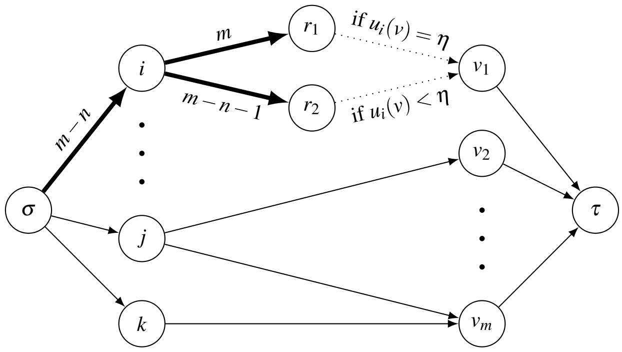

Fig. 4: All the thin arcs have capacity equal 1. Capacity of thick arcs is shown next to them. The condition for dotted arcs to be included is shown next to them.

Proof If $m < n$ then necessarily some agent gets nothing and so as not to create envy each agent has to get a bundle with disutility 0. Since preferences are strict, this is only possible if each agent gets at most one vertex, namely the one where she has disutility 0. We can easily verify whether this happens by matching techniques.

Now let us proceed to the case when agents receive positive disutility. We shall in turn check for each agent i whether she can be the central agent. To this end, order all the vertices in V according to i's disutility increasingly (we might possibly rename the vertices in G)

$$u_i(v_1) < \cdots < u_i(v_{m-n+1}) < u_i(v_{m-n+2}) < \cdots < u_i(v_m) \tag{3}$$

and notice that i will receive exactly $m-n+1$ vertices. Moreover, i must receive the bundle $\{v_1, v_2, \ldots, v_{m-n+1}\}$ that because otherwise another agent, say j, will receive one of these chores (and only this one) and i will envy j. The necessary conditions a central agent i must therefore fulfill are:

(i) vertex ω must be among the vertices $v_1, v_2, \ldots, v_{m-n+1}$ and
(ii) $\sum_{k=1}^{m-n+1} u_i(v_k) \leq u_i(v_{m-n+2})$

These conditions imply that when assigned the bundle $X = \{v_1, v_2, \ldots, v_{m-n+1}\}$, agent i will not envy any other agent. We still have to ensure no envy among other agents.

Let us construct the bipartite graph $H = (Z, Z', L)$ with $Z = N \backslash \{i\}$, $Z' = V \backslash X$ and $\{j, v\} \in L$ if the following inequality is fulfilled:

$$u_j(v) \leq min\{\{u_j(v'); v' \in Z'\}, u_j^{add}(X)\}. \tag{4}$$

Finally, there exists in $\mathscr{I}$ an envy-free assignment where agent i is the central agent if and only if H admits a perfect matching M. Namely, agent i is assigned bundle X and the other agents chores according to M. □

The method described in the proof of the previous theorem can easily be adapted also for the maximum based disutility aggregation. The only difference follows from using maximum (and hence the disutility of just one vertex) in the computation of the disutility of agent's bundle instead of the sum.

Theorem 11 MAX-EF-CCD *is solvable in polynomial time if G is a star and the preferences of agents are strict on chores.*

Proof When checking whether agent i can be central, after we permute the vertices of G according to i's disutility increasingly, it is easy to see that agent i must receive her $m-n+1$ most favorite chores, and it suffices to check just condition (i) to prevent her to envy another agent. Further, in the construction of the bipartite graph H condition (4) should be replaced by

$$u_j(v) \leq min\{\{u_j(v'); v' \in Z'\}, u_j^{max}(X)\} \tag{5}$$

and the rest of the proof follows. □

6.2 Hard cases

When the underlying graph is a star, the only intractable problems concern envy-freeness and equitability, when the disutilities are additive, as we will see now.

Theorem 12 ADD-EF-CCD *is NP-complete even if the underlying graph G is a star.*

Proof To prove the NP-completenes, we provide a reduction from (2,2)-E3-SAT. So we start by a boolean formula F.

We construct an instance of ADD-EF-CCD with $m = 7s+2$ chores and $n = 2s+t+2$ agents defined as follows. The set of chores is $V = Y \cup Z \cup \{\omega, d\}$, where ω denotes the center of the star G, $Y = \bigcup_{i=1}^{s}\{y_i, \bar{y}_i, \tilde{y}_i\}$ are variable chores and $Z = \bigcup_{j=1}^{s}\{z_j^1, z_j^2, \bar{z}_j^1, \bar{z}_j^2\}$ are literal chores. Observe that each literal $\ell \in L$ has its corresponding chore which will be denoted by $z(\ell)$.

The set of agents is $N = B \cup P \cup Q \cup \{e, r\}$, where $B = \{b_1, \dots, b_t\}$ are clause agents and $P = \{p_1, \dots, p_s\}$, $Q = \{q_1, \dots, q_s\}$ are variable agents.

The disutilities are defined as follows.
If $a = b_i \in B$ is a clause agent then:

$$u_{b_i}(v) = \begin{cases} 0 & \text{if } v = d \text{ or } v = z(\ell) \text{ for some } \ell \in L_i \\ 1/(7s-2) & \text{otherwise.} \end{cases}$$

If $a = p_i \in P$ then:

$$u_{p_i}(v) = \begin{cases} 0 & \text{if } v = y_i \text{ or } v = \bar{z}_i^j \text{ for some } j \in \{1,2\} \\ \varepsilon & \text{if } v = \tilde{y}_i \text{ or } v = d \\ (1-\varepsilon)/(7s-3) & \text{otherwise,} \end{cases}$$

where ε is such that $\varepsilon < (1-\varepsilon)/(7s-3)$. Notice that this inequality implies $\varepsilon < 1/(7s-2)$.
If $a = q_i \in Q$ then:

$$u_{q_i}(v) = \begin{cases} 0 & \text{if } v = \bar{y}_i \text{ or } v = z_i^j \text{ for some } j \in \{1,2\} \\ \varepsilon & \text{if } v = \tilde{y}_i \text{ or } v = d \\ (1-\varepsilon)/(7s-3) & \text{otherwise.} \end{cases}$$

If $a = e$ then:

$$u_e(v) = \begin{cases} 1/(s+1) & \text{if } v = d \text{ or } v = \tilde{y}_i \text{ for some } j \in \{1,\dots,s\} \\ 0 & \text{otherwise.} \end{cases}$$

If $a = r$ then:

$$u_r(v) = \begin{cases} 0 & \text{if } v = d \\ 1/(7s+1) & \text{otherwise.} \end{cases}$$

Assume first that f is a truth assignment that satisfies all clauses in C. We construct from f a valid assignment π of chores as follows. For each variable x_i, if x_i is `true`, assign $\tilde{y}_i$ and $\bar{y}_i$ to p_i and q_i respectively, and if x_i is `false`, assign y_i and $\tilde{y}_i$ to p_i and q_i. Furthermore, for each clause c_i pick literal $\ell \in L_i$ which is `true` and assign $z(\ell)$ to b_i. Finally, assign d to r, and the remaining chores to e.

Now we check that this allocation is envy-free. First, no agent whose disutility in π is equal 0 can envy any other agent. This is clearly the case of agents r and agents in B. Agent e receives many chores, but not chore d and no chore of the form $\tilde{y}_i$ (as these are assigned either to agent p_i or to agent q_i), so her disutility is also equal to 0. It remains to show that variable agents in $P \cup Q$ do not envy. Take agent p_i. She is assigned either chore y_i (and her disutility is thus 0, and so we do not need to deal with this case any more) or chore $\tilde{y}_i$, for which she has disutility ε. Therefore she could envy an agent who receives only chores from the set $P_i^T = \{y_i, \bar{z}_i^1, \bar{z}_i^2\}$. However, this happens when variable x_i is `true` in f. In this case, all the chores from P_i^T with many other chores, including ω, are assigned to agent e. This means $u_{p_i}(\pi(e)) > \varepsilon$ and so agent p_i does not envy anybody. The argument for agents in Q is similar.

Conversely, suppose that there is a valid assignment π of chores such that no agent envies another one. Since $\tilde{y}_i$ provides a non-zero disutility to every agent, no agent can receive an empty bundle; otherwise each agent that receives chore $\tilde{y}_i$ will be envious. Furthermore, only the central agent can receive strictly more than one chore, and, as a consequence, this central agent must receive exactly $5s - t + 1$ chores (otherwise some chores will be left unassigned). We show by contradiction that the central agent should be e. Assume by contradiction that the central agent is not e. This central agent will receive a bundle of chores of size $5s - t + 1$, and since the number of chores for which an agent has disutility at most ε is never larger than 5 (except for agent e for which it is $6s + 1$), then at least two chores with the highest disutility are assigned to the central agent. Hence the central agent will envy any other agent receiving just one chore, a contradiction. Therefore, only e can be the central agent.

Suppose that e receives a non-zero disutility. Then she will envy any other agent receiving either one chore other than d or one $\tilde{y}_i$ or no chore at all. It thus means that d and each chore $\tilde{y}_i$ must be assigned to some other agents.

If d is not assigned to r then r will envy the agent who receives it. Hence, d is assigned to r in π. This in turn implies that each agent b_i should receive chore $z(\ell)$ for some $\ell \in L_i$. This also implies that each variable agent should receive a chore for which she has disutility 0 or ε, which means that chore $\tilde{y}_i$ is assigned either to agent p_i or to q_i.

We now construct truth assignment f as follows. If $\tilde{y}_i$ is assigned to p_i in π then set x_i to be `true`, and otherwise (i.e. $\tilde{y}_i$ is assigned to q_i) set x_i to be `false`. We will show that f satisfies each clause c_i. Let ℓ be the literal of c_i such that $z(\ell)$ is assigned to b_i. Assume that ℓ is a positive literal of variable x_j (the negative case can be treated in a similar way). If $\tilde{y}_j$ is assigned to agent q_i then she will envy agent b_i, leading to a contradiction. Therefore, $\tilde{y}_i$ is assigned to agent p_i, x_i is set to `true` and clause c_i is `true`. □

Theorem 13 MAX-EF-CCD *is* NP*-complete even if the underlying graph G is a star.*

Proof The reduction is almost the same as the one presented for Theorem 12 except that the disutility provided by chore ω (the center of the star) is set to 1 for all agents except for agent e, and the disutility of agent e for each chore for which she has disutility $1/(s+1)$ is set to 1. Note first that in that case, no agent except for agent e can receive chore ω without envying the other agents since ω is the only chore providing a disutility of 1. Therefore, in any envy free allocation chore ω is assigned to agent e. It is easy to check that all the other arguments provided in the proof of Theorem 12 hold in this new construction using max aggregator. □

The equitability criterion also leads to an NP-complete problem.

Theorem 14 ADD-EQ-CCD *is* NP*-complete if G is a star.*

Proof We shall provide a polynomial reduction from the following version of the NP-complete problem PARTITION (Garey and Johnson 1979), Problem SP12. The symbol $[p]$ denotes the set $\{1,2,\ldots,p\}$.

Instance $\mathscr{J}$: A set $\{a_i,b_i;i\in[p]\}$ of integers such that $\sum_{i\in[p]}(a_i+b_i)=2K$.
Question: Does there exist a partition (P,P') of $[p]$ such that $\sum_{i\in P}a_i+\sum_{i\in P'}b_i=K$?

Let us construct an instance $\mathscr{I}$ of ADD-EQ-CCD as follows. The set of chores is $V=\{\omega,d\}\cup V'$, where $V'=\{v_i,w_i;i\in[p]\}$. Chore ω is the center of the star, the other chores are its leaves. The set of agents is $N=\{j_0,j_{p+1}\}\cup N'$, where $N'=\{j_i;i\in[p]\}$. Note that there are $2p+2$ chores and $p+2$ agents. The disutilities of agents are as follows and it can be easily checked that they are all normalized.

$$u_{j_0}(v)=\begin{cases}1/6 & \text{for } v=\omega\\ 1/2 & \text{for } v=d\\ a_i/(6K) & \text{for } v=v_i;i\in[p]\\ b_i/(6K) & \text{for } v=w_i;i\in[p]\end{cases}$$

$$u_{j_{p+1}}(v)=\begin{cases}1/3 & \text{for } v=d\\ 2/(6p+3) & \text{otherwise}\end{cases}$$

and, finally, for $i\in[p]$

$$u_{j_i}(v)=\begin{cases}1/3 & \text{for } v\in\{\omega,v_i,w_i\}\\ 0 & \text{otherwise}\end{cases}$$

Now suppose that (P,P') is a partition witnessing that $\mathscr{J}$ is a yes instance; let us define a valid assignment in the following way. $\pi(j_0)=\{\omega\}\cup\{v_i,i\in P\}\cup\{w_i,i\in P'\}$ and $\pi(j_{p+1})=\{d\}$. Further, $\pi(j_i)=\{v_i\}$ if $i\in P'$ and $\pi(j_i)=\{w_i\}$ if $i\in P$. It is easy to see that all vertices are assigned, each agent $j\neq j_0$ has one leaf with disutility $1/3$ and the bundle of j_0 has disutilty also $1/3$ thanks to the properties of partition (P,P').

Conversely, suppose that $\mathscr{I}$ admits a valid equitable assignment π such that each agent gets the same disutility equal to η. First, let us realize that $\eta\neq 0$, as the agent who receives the central vertex ω has a positive disutility. Further, η cannot be strictly greater than $1/3$. Namely, as exactly one agent can be assigned ω (and hence more than one chore), we would not be able to give to each agent in N' a piece with disutility greater than $1/3$. Thus, each agent in N' can receive only one chore and this means that their disutility is either 0 or $1/3$. Therefore $\eta=1/3$. This also immediately implies that $\pi(j_{p+1})=\{d\}$. Furthermore, the central agent is j_0 since the central agent should receive $p+1$ chores and j_0 is the only agent who can have a disutility of at most $1/3$ for such a large bundle of chores. Therefore, each agent $j_i\in N'$ receives either v_i or w_i and agent j_{p+1} receives chore d. This means that $\pi(j_0)=\{\omega\}\cup\{v_i;i\in P\}\cup\{w_i;i\in P'\}$ where (P,P') is a partition of $[p]$; moreover, as disutility j_0 derived from $\pi(j_0)$ is equal to $1/3$, we must have

$$\sum_{i\in P}a_i/(6K)+\sum_{i\in P'}b_i/(6K)=1/6$$

which implies

$$\sum_{i \in P} a_i + \sum_{i \in P'} b_i = K$$

and hence (P, P') is a partition of $[p]$ verifying that $\mathscr{J}$ is a yes instance of PARTITION. $\square$

7 Conclusion and open problems

In this paper we studied the computational complexity of the problem of finding a fair allocation of nondisposable undesirable items (chores) with the additional requirement that each agent has to receive a bundle of chores that is connected in the graph representing the relationship between items.

We have demonstrated that the chore division problems and their corresponding "dual" fair division problems do not necessarily have solutions that directly translate from one context to another, moreover, the computational complexity of the corresponding problems can differ.

We proposed polynomial algorithms for some existence problems and showed that other problems are NP-complete. Moreover, our construction for paths even leads to multiplicative inapproximability results.

Notice that Suksompong (2017) considered fair allocations on paths with contiguous bundles that are approximately fair up to an additive factor. His constructions could be, with very minor modifications, applied also in the chore-division problem. However, we do not know how to construct additively approximately fair valid allocations for the other simple graph, the star.

Other natural open questions can be thought of. Is there any graph structure that could separate the polynomial problems from the intractable ones for various fairness criteria? When no valid fair allocation exists, one can think of some relaxations of the connectivity constraints. One could for example ask that the share of each agent should consist of not more than k disconnected pieces or that each diameter should be bounded.

Further, we have omitted the recently introduced fairness criteria maximin share guarantee and envy-freeness up to one good. We believe that they may lead to some more interesting results.

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