

# 12

## Fair Allocation of Indivisible Goods

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As introduced in Chapter 11 (Thomson, 2015), *Fair Allocation* (or *Fair Division*) refers to the general problem of fairly dividing a common resource among agents having different —and sometimes antagonistic— interests in the resource. But under this general term one can actually gather a cluster of very different problems, all calling for different solution concepts: after all, one can easily figure out that we cannot allocate a set of objects like a bicycle, a car or a house like we allocate pieces of land.

In this chapter, we will focus on fair division of *indivisible goods*. In other words, the resource is here a set  $\mathcal{O} = \{o_1, \dots, o_p\}$  of *objects* (that may also be called *goods* or *items*). Every object must be allocated *as is*, that is, an object loses its value if it is broken or divided into pieces to be allocated to several individuals. This assumption makes sense in a lot of real-world situations, where indivisible goods can be for example physical objects such as houses or cars in divorce settlements, or “virtual” objects like courses to allocate to students (Othman et al., 2010) or Earth observation images (Lemaître et al., 1999). Moreover, we assume in this chapter that the objects are *non-shareable*, which means that the same item cannot be allocated to more than one agent. This assumption seems to be questionable when the objects at stake are rather non-rival, that is, when the consumption of one unit by an agent will not prevent another one from having another unit (what we referred to as “virtual” objects). In most applications, such non-rival objects are available in limited quantity though (*e.g.* number of attendants in a course). This kind of problems can always be modeled with non-shareable goods by introducing several units of the same good.

What mainly makes fair division of indivisible goods specific, if not more difficult, is that classical fairness concepts like envy-freeness or proportionality are sometimes unreachable, unlike in the divisible (*a.k.a.* cake-cutting) case. As an illustration of this difference, consider a (infinitely divisible) piece of land which has to be split

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among two individuals, Alice and Bob. One classical way to proceed (see Chapter 13, Procaccia, 2015) is to let Alice propose a cut, and then let Bob take the share he prefers. If Alice acts rationally, she will cut the land into two pieces of the same value to her (if she acts differently she may end up with a worse piece), and hence will not envy Bob’s piece. Such an *envy-free* allocation is always reachable with a divisible resource, but computing this allocation may require an unbounded number of cuts, as we will see in Chapter 13, Section 13.4 (Procaccia, 2015). Even in the presence of indivisible items, the use of a special divisible resource (money) allows to “transfer utility” and suffices to guarantee this existence (Beviá, 1998). This is not the case when only indivisible goods are available: if, in the extreme case, there is a single good and two agents, one of the two will obviously be despoiled. Worse, in the general case, figuring out for a given instance whether such a fair solution even exists can be very complex (see Section 12.3).

To circumvent this issue, some authors reintroduce some divisibility in the process, either by relaxing the integrity of some goods and allocating *fractions* of these goods, as in the Adjusted Winner procedure proposed by Brams and Taylor (2000) and explained in Section 12.4, or by using *money* as an *ex-post* compensation for despoiled agents. When these kinds of solutions are not available, some authors (among which Brams et al., 2014) propose to relax the assumption that all the objects should be allocated. Another option is to relax our fairness requirement and focus on weaker solution concepts. These two last options correspond to the two possible solutions to the classical fairness versus efficiency trade-off (Section 12.2).

We cannot conclude this overview of distinctive features of indivisible goods without mentioning *preferences*. Preferences are at the heart of fair division, because fairness is often related to what the agents prefer to get from the allocation, may it be what they need, or what they just would like to have. To be able to compare two different allocations, we should first be able to figure out how the agents at stake evaluate their shares. This may come down to answering questions like: “does Alice prefer the bike and the boat together or the car alone?”. While the number of shares to compare is finite, this number is huge, and makes the explicit representation of agents’ preferences unrealistic. Concise preference representation is yet not out of reach, and can be achieved at the price of restricting assumptions—like additivity— or increased complexity (see Section 12.1). However, as can be seen in Chapter 13 (Procaccia, 2015), such preference representation languages do not really transpose to the divisible case, which makes the design of *centralized* one-shot procedures less relevant to this case. This may explain why many works in fair division of indivisible goods focus on complexity and algorithmic issues of centralized allocation procedures (see Section 12.3), while the literature in cake-cutting is more concerned with the design of interactive protocols for fair division. There are nevertheless prominent protocols for the allocation of indivisible items, we review some of them in Section 12.4.

*Preliminary definitions.* We will now introduce a few formal definitions which will be used all along the chapter. In this chapter,  $N = \{1, \dots, n\}$  will be a set of  $n$  *agents*, and  $\mathcal{O} = \{o_1, \dots, o_p\}$  a set of  $p$  (indivisible, non-shareable) *objects*. Each subset  $\mathcal{S}$  of  $\mathcal{O}$  is called a *bundle*. In the following, we will sometimes write  $o_1o_2o_3$  as a shortcut for bundle  $\{o_1, o_2, o_3\}$ . An *allocation* is a function  $\pi : N \rightarrow 2^{\mathcal{O}}$  mapping each agent to the bundle she receives, such that  $\pi(i) \cap \pi(j) = \emptyset$  when  $i \neq j$  since the items cannot be shared. The subset of objects  $\pi(i)$  will be called agent  $i$ 's *bundle* (or *share*). When  $\bigcup_{i \in N} \pi(i) = \mathcal{O}$ , the allocation is said to be *complete*. Otherwise, it is *partial*. The set of all allocations is denoted  $\Pi$ .

Following Chevaleyre et al. (2006), a *MultiAgent Resource Allocation setting* (MARA setting for short) denotes a triple  $(N, \mathcal{O}, R)$ , where  $N$  is a finite set of agents,  $\mathcal{O}$  is a finite set of indivisible and non-shareable objects, and  $R$  is a sequence of  $n$  *preference relations* on the bundles of  $\mathcal{O}$ . The notion of “preference relation” has to be properly defined, which is not straightforward, and is the topic of the entire next section.

## 12.1 Preferences for Resource Allocation Problems

In order to allocate the indivisible goods properly to the agents, the community (or the benevolent arbitrator acting on behalf of it) needs to take into account the agents' wishes about the goods they want to receive. In other words, one has to be able to compare the different allocations based on the preferences the agents have on what they receive.<sup>1</sup> As we have seen in the introduction, the particular structure of the set of allocations is the main distinctive feature of resource allocation of indivisible goods, that makes the expression of preferences and the resolution of this kind of problems particularly difficult from a computational point of view.

### 12.1.1 Individual preferences: from objects to bundles

The minimal and most natural assumption we can reasonably make on the agents is that they are at least able to compare each pair of individual items, just like voters are able to compare each pair of candidates in an election setting (see Chapter 2, Zwicker, 2015). In other words, we can minimally assume that each agent  $i$  is equipped with a preorder  $\succeq_i$  on  $\mathcal{O}$ . Two further assumptions that are commonly made are that this relation is:

- either a *linear order*  $\triangleright_i$ , which basically means that each agent is able to rank each item from the best to the worst, with no ties allowed (this is the classical preference model in voting theory);

<sup>1</sup> We assume that the agents only care about what they receive, and not what the others receive. This assumption of *non-exogenous preferences* is commonly made in the context of fair division.

- or represented by a *utility function*  $w_i : \mathcal{O} \rightarrow \mathbb{F}$ , mapping each object to a score taken from a numerical set (that we will assume to be  $\mathbb{N}$ ,  $\mathbb{Q}$  or  $\mathbb{R}$  for the sake of simplicity).

Unlike in voting theory, ranking items is generally not enough to provide valuable information about the agents' preferences concerning different allocations. Consider for example a setting where four objects  $\{o_1, o_2, o_3, o_4\}$  have to be allocated to two different agents. Suppose that agent 1 ranks the objects as follows:  $o_1 \triangleright o_2 \triangleright o_3 \triangleright o_4$ . Does it mean that she would prefer an allocation that would give her  $o_1$  and  $o_4$  to an allocation that would give her  $o_2$  and  $o_3$ ? Or an allocation that would give her  $o_1$  to an allocation that would give her  $o_2$  and  $o_4$ ?

The technical problem that lies behind this kind of questions is the problem of *lifting* the preference relation  $\succeq$  (or the utility function  $w$ ) on individual objects to a preference relation  $\succsim$  (or a utility function  $u$ ) on *bundles* of objects.<sup>2</sup> There are two possible ways of doing it:

1. either by automatically lifting preferences to bundles of objects using some natural assumptions;
2. or by asking the agents to rank not only the individual objects but also the bundles of objects.

### 12.1.2 Additive preferences

The first approach has been considered by several authors, either in economics (Brams and King, 2005; Herreiner and Puppe, 2009) or in computer science (Lipton et al., 2004; Bansal and Sviridenko, 2006; Bouveret et al., 2010). These works are usually based on a cardinal property and its ordinal counterpart, which can be reasonably assumed in many resource allocation contexts:

**Definition 12.1** (Modularity) A utility function  $u : 2^{\mathcal{O}} \rightarrow \mathbb{F}$  is *modular* if and only if for each pair of bundles  $(\mathcal{S}, \mathcal{S}')$ , we have  $u(\mathcal{S} \cup \mathcal{S}') = u(\mathcal{S}) + u(\mathcal{S}') - u(\mathcal{S} \cap \mathcal{S}')$ .

An equivalent definition is that for each bundle  $\mathcal{S}$ ,  $u(\mathcal{S}) = u(\emptyset) + \sum_{o \in \mathcal{S}} u(\{o\})$ .

If we further assume that the utility of an agent for the empty set ( $u(\emptyset)$ ) is 0, then we can compute the utility of an agent for each bundle of objects  $\mathcal{S}$  by just summing the scores given by this agent to each individual object in  $\mathcal{S}$ . In this case, the utility function is said to be *additive*. This is one of the most classical settings in fair division of indivisible goods.

Additivity is a very strong property that forbids any kind of synergy between objects. Going back to our previous example with four objects, additivity implies that since agent 1 prefers  $o_1$  to  $o_2$ , she will also prefer  $\{o_1, o_3\}$  to  $\{o_2, o_3\}$ . This makes sense if  $o_3$  is rather uncorrelated to  $o_1$  and  $o_2$ : for example, if  $o_1$  is a voucher for a

<sup>2</sup> The problem of lifting preferences over items to preferences over bundles has actually been studied in depth in social choice theory (Barberà et al., 2004).

train ticket in France,  $o_2$  is a voucher for a night in Paris, and  $o_3$  is a camera, it seems reasonable to assume that my preference on taking the train rather than spending a night in Paris will hold, no matter whether a camera is delivered with the voucher or not. Another way to state it is to say that if in bundle  $\{o_2, o_3\}$   $o_2$  is replaced by a better object (*e.g.*  $o_1$ ), then it makes a better bundle. This feature corresponds to a notion called pairwise-dominance, or *responsiveness* (Barberà et al., 2004), which can be stated formally in a purely ordinal context:  $\forall \mathcal{S} \subset \mathcal{O}$  and all  $o \in \mathcal{S}$  and  $o' \in \mathcal{O} \setminus \mathcal{S}$ ,  $(\mathcal{S} \succ \mathcal{S} \setminus \{o\} \cup \{o'\} \Leftrightarrow \{o\} \succ \{o'\})$  and  $(\mathcal{S} \setminus \{o\} \cup \{o'\} \succ \mathcal{S} \Leftrightarrow \{o'\} \succ \{o\})$ .<sup>3</sup>

Responsiveness is used among others by Brams et al. (2004); Brams and King (2005) to lift preferences defined as a linear order  $\triangleright$  over single objects to a preference relation over bundles of objects of the same cardinality.<sup>4</sup> To be able to compare bundles of different cardinalities, some authors (Bouveret et al., 2010; Brams et al., 2012) add a monotonicity assumption stating that if  $\mathcal{S} \supset \mathcal{S}'$ ,  $\mathcal{S} \succ \mathcal{S}'$ .<sup>5</sup> Responsiveness (in its strict form, or with possible indifferences) plays an important role in fair division under ordinal preferences, because it has an interesting implication. An agent with responsive preferences will always be able to pick unambiguously the object that she prefers among a set, this choice being independent from what she has already received, and what she will receive later on. This property guarantees that some protocols for fair division such as the undercut procedure (see Section 12.4.1) or picking sequences (see Section 12.4.2) work properly. As mentioned above, this property, in its strict form, is also at the basis of a few works (Brams et al., 2004; Brams and King, 2005; Bouveret et al., 2010), the latter having been extended by Aziz et al. (2014) to deal with (responsive) preferences with indifferences.

Note that, interestingly, it can be easily shown that any preference relation  $\succ$  obtained by lifting a linear order  $\triangleright$  over single objects using pairwise dominance and monotonicity can be represented by any additive utility function  $u$  (*i.e.*  $u(\mathcal{S}) > u(\mathcal{S}') \Leftrightarrow \mathcal{S} \succ \mathcal{S}'$ ), as soon as  $u$  is compatible with the linear order (*i.e.*  $u(o) > u(o') \Leftrightarrow o \triangleright o'$ ). However, things are not so simple as soon as indifferences between bundles are allowed: as mentioned by Barberà et al. (2004), additive representability only entails *responsiveness*, but is not equivalent.<sup>6</sup>

### 12.1.3 Beyond additivity

Going back to the previous example, additivity makes sense when the objects at stake are rather unconnected (a train ticket and a camera in the example). However, things are different if the objects are of similar nature or are closely coupled. For

<sup>3</sup> To be precise, in the original definition by Barberà et al. (2004) the comparisons are not strict, but some authors like *e.g.* Brams et al. (2012) use this strict version of responsiveness.

<sup>4</sup> Such a lifting is called the *responsive set extension*.

<sup>5</sup> Monotonicity will be formally introduced on page 12.

<sup>6</sup> Another important property is *extended independence*, which states that for every pair of bundles  $(\mathcal{S}, \mathcal{S}')$ , and every bundle  $\mathcal{S}''$  such that  $(\mathcal{S} \cup \mathcal{S}') \cap \mathcal{S}'' = \emptyset$ , we have:  $\mathcal{S} \succ \mathcal{S}' \Rightarrow \mathcal{S} \cup \mathcal{S}'' \succ \mathcal{S}' \cup \mathcal{S}''$ . Additive representability entails extended independence which in turn entails responsiveness.

example, if  $o_3$  is now a plane ticket for the same day as the train ticket, we can reasonably assume that my preferences will be reversed, since now only the night in Paris is compatible with the plane ticket (so by getting the night and the plane ticket I can enjoy both, whereas by getting the train and plane tickets I will have to drop one of the two). This is a case where additive preferences fail to represent what the agents really have in mind, because there are some *dependencies* between objects. These dependencies (or synergies) can be of two kinds: *complementarity* or *substitutability*. Complementarity occurs when having a group of objects is worth more than the “sum” of their individual values: the agent benefits from using them jointly. Going back to our previous example, the plane ticket and the night in Paris can be considered as complementary (if I am not living in Paris): I can use the plane ticket to fly to Paris, and then spend the night there. Substitutability occurs when objects are of very similar nature and when their use is mutually exclusive. In our example, the plane and the train tickets are exclusive, and thus their joint value is not more than the value of one of the two.

A way to circumvent this problem is to allocate the items by pre-made bundles instead of proposing them individually (just like most shoe retailers sell shoes by pairs, not individually). However, in most cases the preferential dependencies are of subjective nature, and complementary and substitutable items are simply not the same for everyone.<sup>7</sup> In that case, we just cannot do anything else than asking the agents to rank all the possible bundles of objects. As the reader might guess however, the number of possible bundles obviously grows exponentially with the number of objects, which renders the explicit ranking of all bundles simply impossible as soon as the number of objects exceeds 4 or 5. To illustrate this combinatorial blow-up, consider a resource allocation problem with just 16 objects, which seems to be a setting of very reasonable (if not small) size. In such a problem, each agent will have to compare  $2^{16} = 65536$  bundles, which comes down to a tremendous (and unrealistic) amount of work for the agents.

As we can see, the community of agents or the benevolent arbitrator acting on behalf of it faces a dilemma: either restricting the set of expressible preferences to additive ones and hence ruling out the expression any kind of preferential dependencies, or letting the agents compare all pairs of possible bundles and falling in the combinatorial blow-up trap.

#### 12.1.4 Compact preference representation

*Compact preference representation languages* can be seen as a compromise, often made at the price of increased computational complexity. The idea here is to use an intermediate language which can represent the agents’ preferences as closely

<sup>7</sup> For example, a laptop computer and a tablet-PC might be complementary for individuals doing a lot of writing at home (they would need a good keyboard) and a lot of reading while traveling (they would require a lightweight device). For others, these devices might be substitutable.

as possible, while formulas in that language remain as compact as possible. One formula in this language simply represents one preference relation on the bundles of objects. More formally:

**Definition 12.2** (Preference representation language) An ordinal (resp. a cardinal) preference representation language is a pair  $\langle L, I(L) \rangle$  that associates to each set of objects:<sup>8</sup>

- a language  $L(\mathcal{O})$  (*i.e.* a vocabulary and a set of well-formed formulas) —the syntactical part of the language;
- an interpretation  $I(L)(\mathcal{O})$  that maps any well-formed formula  $\varphi$  of  $L(\mathcal{O})$  to a preorder  $\succsim_\varphi$  of  $2^{\mathcal{O}}$  (resp. a utility function  $u_\varphi : 2^{\mathcal{O}} \rightarrow \mathbb{F}$ ) —the semantical part of the language.

A trivial example of preference representation language is the *bundle form*, which can be seen as a form of explicit representation. A formula in this language is just made of a set of pairs  $\langle \mathcal{S}, u_{\mathcal{S}} \rangle$ , where  $\mathcal{S}$  is a bundle of objects, and  $u$  is a non-zero numerical weight. The utility of a given bundle  $\mathcal{S}$  is just  $u_{\mathcal{S}}$  if  $\langle \mathcal{S}, u_{\mathcal{S}} \rangle$  belongs to the set, and 0 otherwise.

One might wonder what in the use of an intermediate language for representing ordinal or numerical preferences makes the representation “compact”. Actually, reconciling (full) expressivity and succinctness is an unsolvable equation for the following reason. If for the sake of example we consider numerical preferences, the number of utility functions from  $2^{\mathcal{O}} \rightarrow \{0, \dots, K-1\}$  is  $K^{2^p}$  (with  $p = |\mathcal{O}|$ ). Following an information-theoretic argument recalled by Cramton et al. (2006), it means that if our language is fully expressive, some utility functions will need at least  $2^p \frac{\ln K}{\ln 2}$  bits to be encoded as a formula, since no encoding of  $t$  bits is able to discriminate more than  $2^t$  words. Hence, compact preference representation is not a matter of representing all preference relations in reasonable (polynomial) size, but just the *interesting* ones, that is the ones that are more likely to correspond to what the agents will naturally express. For example, the bundle form language described above can be considered compact only if it is reasonable to assume that the agents will value positively only a small number of bundles.

*Additivity generalized.* Let us take another example of what we mean by “interesting preference relations”. Consider additive utility functions introduced earlier. Their main advantages are their conciseness (each agent just needs to provide one weight for each object) and their simplicity. However, their annoying drawback is that they are unable to encode even the slightest complementarity or substitutability between objects. On the other hand, allowing any kind of synergy exposes us to the computational blow-up, whereas it is very likely that an agent would be willing to

<sup>8</sup> Preference representation languages can be used more generally to represent a preference relation on any combinatorial set of alternatives. For the sake of simplicity, we choose to restrict the definition here to sets of objects.

express only synergies concerning a limited number of objects (do we really need to give the agents the opportunity of expressing the added value of owning a bundle of 42 objects compared to the values of its proper subsets?). This is the idea behind  $k$ -additive numerical preferences:

**Definition 12.3** ( $k$ -additive preference representation language) A formula in the  $k$ -additive representation language is a set  $\mathcal{B}$  of pairs  $\langle \mathcal{S}, w_{\mathcal{S}} \rangle$ , where  $\mathcal{S} \subseteq \mathcal{O}$  is a bundle of size at most  $k$ , and  $w_{\mathcal{S}}$  is a non-zero numerical weight. Given a formula  $\mathcal{B}$  in this language, the utility of each bundle  $\mathcal{S}$  is defined as:

$$u(\mathcal{S}) = \sum_{\substack{\langle \mathcal{S}', w_{\mathcal{S}'} \rangle \in \mathcal{B} \\ \mathcal{S}' \subseteq \mathcal{S}, |\mathcal{S}'| \leq k}} w_{\mathcal{S}'} \quad (12.1)$$

The weight  $w_{\mathcal{S}}$  represents the added value of  $\mathcal{S}$ , beyond the value of its proper subsets, or in other words, the synergistic potential of  $\mathcal{S}$ .<sup>9</sup> If this number is positive, it means that the objects in  $\mathcal{S}$  work in complementarity, if it is negative, these objects are probably substitutable. A utility function  $u$  whose weights of size 2 or more are positive (resp. negative) has the *supermodularity* (resp. *submodularity*) property. In other words, it holds that that  $u(\mathcal{S} \cup \mathcal{S}') \geq u(\mathcal{S}) + u(\mathcal{S}') - u(\mathcal{S} \cap \mathcal{S}')$  (resp.  $u(\mathcal{S} \cup \mathcal{S}') \leq u(\mathcal{S}) + u(\mathcal{S}') - u(\mathcal{S} \cap \mathcal{S}')$ ).

**Example 12.4** Let  $\mathcal{O} = \{o_1, o_2, o_3, o_4\}$  be a set of objects, and let  $u$  be the  $k$ -additive utility function defined from the following set of weights:  $\langle o_1, 2 \rangle$ ,  $\langle o_2, 2 \rangle$ ,  $\langle o_1 o_2, -2 \rangle$ ,  $\langle o_3 o_4, 10 \rangle$ . All other bundles have weight zero. We have for example  $u(o_1) = u(o_2) = 2$ , and  $u(o_1 o_2) = 2 + 2 - 2$ , which is 2 as well. This probably means that objects  $o_1$  and  $o_2$  are *substitutes* (having both does not give more utility than having just one). On the contrary,  $o_3$  and  $o_4$  alone are useless ( $u(o_4) = u(o_3) = 0$ ), but having them together is interesting ( $u(o_3 o_4) = 10$ ), which means that they act as *complementary objects*. We can also notice that  $u$  is neither modular, nor sub- or supermodular.

The succinctness of the language is ensured by the parameter  $k$ , that bounds the size of formulas representing our utility functions to  $\sum_{i=0}^k \binom{p}{i} = O(p^k)$ . This parameter  $k$  can be seen as a value that represents the trade-off between full expressivity (and formulas of potentially exponential size) if  $k = p$  and limited expressivity (and formulas of linear size), *i.e.* additive functions, if  $k = 1$ .

*Graphical models.* Interestingly, the  $k$ -additive preference representation coincides, for the special case of bundle combinatorial spaces we have to deal with in resource allocation problems, with a more general preference representation language: *GAI-nets* (Bacchus and Grove, 1995; Gonzales and Perny, 2004). The language

<sup>9</sup> These weights are also called *Möbius masses* in the context of fuzzy measures, where this kind of representation is extensively used (see *e.g.* Grabisch, 1997).



of GAI-nets is a *graphical model* for preference representation. Graphical models are a family of knowledge representation languages, which have been introduced decades ago in the context of uncertainty (*e.g.* influence diagrams, see Howard and Matheson, 1984) probabilistic modeling (*e.g.* Bayesian networks, see Pearl, 1988), constraint satisfaction (Montanari, 1974) or valued constraint optimization (Schiex et al., 1995). In all these contexts, graphical models are based upon the same components: (i) a graphical component describing directed or undirected dependencies between variables; (ii) a collection of local statements on single variables or small subsets of variables, compatible with the dependence structure. In the particular case of GAI-nets, the preferential (in)dependence notion upon which this language is built is *generalized additive independence* (GAI), introduced by Fishburn (1970), further developed by Keeney and Raiffa (1976) in the context of multiattribute decision making. The  $k$ -additive representation introduced earlier can be seen as a GAI representation on a bundle space, where the size of the local relations (synergies) is explicitly bounded by  $k$ , and with no associated graphical representation.

GAI-nets are not the only graphical model for compact preference representation. Boutilier et al. (1999, 2004) have developed a very powerful and popular preference representation language: CP-nets. Unlike GAI-nets, CP-nets are dedicated to the representation of *ordinal preferences*. Here, the graphical structure describes the (directed) preferential dependencies between variables. The local statements, for each variable, describe the agents' ordinal preferences on the values of the variable's domain, given all the possible combinations of values of its parents (hence "CP" standing for "Conditional Preferences"), and all other things being equal (*ceteris paribus*).

CP-nets have been extended to a family of preference representation languages with different features (see *e.g.* Brafman et al., 2006, for TCP-nets, Boutilier et al., 2001, for UCP-nets and Wilson, 2004, for CP-theories). One of these languages, CI-nets (Bouveret et al., 2009) is especially dedicated to the representation of ordinal preferences on sets of objects, hence well-suited to fair division problems. Formally, a CI-net  $\mathcal{N}$  is a set of CI statements (where CI stands for Conditional Importance) of the form  $\mathcal{S}^+, \mathcal{S}^- : \mathcal{S}_1 \triangleright \mathcal{S}_2$  (where  $\mathcal{S}^+, \mathcal{S}^-, \mathcal{S}_1$  and  $\mathcal{S}_2$  are pairwise-disjoint subsets of  $\mathcal{O}$ ). The informal reading of such a statement is: "if I have all the items in  $\mathcal{S}^+$  and none of those in  $\mathcal{S}^-$ , I prefer obtaining all items in  $\mathcal{S}_1$  to obtaining all those in  $\mathcal{S}_2$ , all other things being equal (*ceteris paribus*)."  
Formally, the interpretation of a CI-net  $\mathcal{N}$  is the smallest monotonic strict partial order  $\succ$  that satisfies each CI-statement in  $\mathcal{N}$ , that is, for each CI-statement  $\mathcal{S}^+, \mathcal{S}^- : \mathcal{S}_1 \triangleright \mathcal{S}_2$ , we have  $\mathcal{S}' \cup \mathcal{S}^+ \cup \mathcal{S}_1 \succ \mathcal{S}' \cup \mathcal{S}^- \cup \mathcal{S}_2$  as soon as  $\mathcal{S}' \subseteq \mathcal{O} \setminus (\mathcal{S}^+ \cup \mathcal{S}^- \cup \mathcal{S}_1 \cup \mathcal{S}_2)$ .

**Example 12.5** Let  $\mathcal{O} = \{o_1, o_2, o_3, o_4\}$  be a set of objects, and let  $\mathcal{N}$  be the CI-net defined by the two following CI-statements:  $S1 = (o_1, \emptyset : o_4 \triangleright o_2 o_3)$ ;  $S2 = (\emptyset, o_1, : o_2 o_3 \triangleright o_4)$ .

From  $\mathcal{N}$ , we can deduce for example that  $o_1 o_4 \succ o_1 o_2 o_3$  ( $S1$ ) and  $o_2 o_3 \succ o_4$  ( $S2$ ).

We can notice that obviously  $\succ$  is not responsive, as having  $o_1$  or not in the bundle reverses the preference between  $o_2o_3$  and  $o_4$ .

CI-nets are a quite natural way of expressing preferences on subsets of objects. However, as we shall see later on, computational complexity is the price to pay for this cognitive relevance. A strict subset of this language, SCI-nets, that coincides with responsive monotonic preferences, have been further investigated from the point of view of fair resource allocation (Bouveret et al., 2010).

*Logic-based languages.* Another family of compact representation languages, which, unlike  $k$ -additive representation or graphical models, is not based on limited synergies, is the family *logical languages*. As we will see, propositional logic is well-suited to represent preferences on subsets of objects, because any set of subsets of objects can be represented (often compactly) by a propositional formula.

In the following, given a set of objects  $\mathcal{O}$ , we will denote by  $\mathcal{L}_{\mathcal{O}}$  the propositional language built upon the usual propositional operators  $\wedge$ ,  $\vee$  and  $\neg$ , and one propositional variable for each object in  $\mathcal{O}$  (for the sake of simplicity we use the same symbol for denoting the object and its associated propositional variable). Each formula  $\varphi$  of  $\mathcal{L}_{\mathcal{O}}$  represents a *goal* that an agent is willing to achieve. From any bundle  $\mathcal{S}$  we can build a logical interpretation  $\mathcal{I}(\mathcal{S})$  by setting all the propositional variables corresponding to an object in  $\mathcal{S}$  to  $\top$  and the other to  $\perp$ . A bundle  $\mathcal{S}$  *satisfies* a goal  $\varphi$  (written  $\mathcal{S} \models \varphi$ ) if and only if  $\mathcal{I}(\mathcal{S}) \models \varphi$ . A goal  $\varphi$  thus stands for a compact representation of the set of all bundles that satisfy  $\varphi$ .

**Example 12.6** Let  $\mathcal{O} = \{o_1, o_2, o_3\}$  be a set of objects. The goal  $\varphi = o_1 \vee (o_2 \wedge o_3)$  is a compact representation of the set of bundles  $\{o_1, o_1o_2, o_1o_3, o_2o_3, o_1o_2o_3\}$ .

The most obvious way of interpreting a goal as a preference relation is to consider that the agent is only happy if the goal is satisfied, and unhappy otherwise. This leads to a dichotomous preference relation  $\succsim^\varphi$  that is defined as follows: for each pair of bundles  $\langle \mathcal{S}, \mathcal{S}' \rangle$ , we have  $\mathcal{S} \succ^\varphi \mathcal{S}'$  if and only if  $\mathcal{S} \models \varphi$  or  $\mathcal{S}' \not\models \varphi$ .

This approach is not very subtle: the agent is not even able to express the tiniest preference between two different objects she both desires. We can do better:

- The first idea is to allow an agent to express several goals (a *goal base*) at the same time. Counting the number of goals satisfied by a given bundle for example gives a good idea of how interesting the bundle is for an agent.
- The second idea is to further allow an agent to *prioritize* the goals of her goal base (bundles are then evaluated in terms of the higher priority goal they satisfy).
- A third idea is that, beyond prioritizing her goal, an agent gives a weight (or a score) to each of them. This idea leads to the weighted logic-based preference representation language that is described below.

**Definition 12.7** (Weighted logic-based preference representation language) A

formula in the weighted logic-based representation language is a set  $\Delta$  of pairs  $\langle \varphi, w_\varphi \rangle$ , where  $\varphi$  is a well-formed formula of the propositional language  $\mathcal{L}_{\mathcal{O}}$ , and  $w_\varphi$  is a non-zero numerical weight.

Given a formula  $\Delta$  in this language, the utility of each bundle  $\mathcal{S}$  is defined as:

$$u(\mathcal{S}) = \sum_{(\varphi, w_\varphi) \in \Delta \mid \mathcal{S} = \varphi} w_\varphi \quad (12.2)$$

Note that in Equation (12.2) any other aggregation operator can be used, such as for example the maximum that selects only the highest weight among the satisfied goals (Bouveret et al., 2005).

**Example 12.8** Let  $\mathcal{O} = \{o_1, o_2, o_3\}$  be a set of objects. The goal  $\Delta = \{\langle o_1 \vee o_2, 1 \rangle, \langle o_2 \wedge o_3, 2 \rangle\}$  is a compact representation of the utility function:

$\mathcal{S}$	$\emptyset$	$o_1$	$o_2$	$o_3$	$o_1 o_2$	$o_1 o_3$	$o_2 o_3$	$o_1 o_2 o_3$
$u(\mathcal{S})$	0	1	1	0	1	1	3	3

The interested reader can refer to Lang (2004) for an extensive survey on logic-based preference representation languages. Coste-Marquis et al. (2004) and Uckelman (2009) provide some detailed results about the expressivity, succinctness and computational complexity of these languages.

*Bidding languages.* We conclude this introduction about compact preference representation for resource allocation by discussing a domain closely related to ours: *auctions*. Auctions are only distinguished from a general resource allocation problem with indivisible goods by the fact that money is at the heart of the evaluation scale (utility here actually represents the amount an agent is ready to pay to obtain some object), and that auctioneers do not care about end-state fairness issues in general. Beyond these “ethical” differences, nothing formally distinguishes an auction setting from a general resource allocation problem.

In the classical auction setting, buyers (or sellers if we deal with reverse auctions) can only bid on individual objects. As a result, the same expressivity problem as the one aforementioned for additive preferences occurs: a bidder is simply unable to express her preference in a proper way if she has preferential dependencies between the objects to be sold. This issue has led Rassenti et al. (1982) to define a new auction setting, where bidders can actually bid on *bundles* of items, instead of just individual items: *combinatorial auctions* (Cramton et al., 2006). To overcome the combinatorial blow-up caused by the explicit representation of set functions, this community has developed its own stream of compact preference representation languages: *bidding languages*. We will not describe these languages here, but the interested reader can refer to the book by Cramton et al. (2006) and especially its chapter about bidding languages (Nisan, 2006) for more information.

*About monotonicity.* Beyond all these preference representation approaches, a property that is often taken for granted in most fair division contexts is monotonicity:

**Definition 12.9** (Monotonicity) A preference relation  $\succsim$  on  $2^{\mathcal{O}}$  is *monotonic* (resp. *strictly monotonic*) if and only if  $\mathcal{S} \subsetneq \mathcal{S}' \Rightarrow \mathcal{S} \succsim \mathcal{S}'$  (resp.  $\mathcal{S} \prec \mathcal{S}'$ ).

Monotonicity formalizes the fact that all the objects have a positive value for each agent, and that the “more” objects an agent receives, the “happier” she will be. Going back to our previous example with four objects, monotonicity implies here that our agent prefers for example  $o_1$  and  $o_2$  together to  $o_1$  alone. This assumption is very natural as long as we are dealing with “positive” objects or “negative” ones, such as tasks or chores (reversing the inequality in this case), but not mixing the two. For most typical compact preference representation languages, the monotonicity assumption has a very natural translation into a simple property on the formulas. For example, for numerical modular preferences, monotonicity is equivalent to the positivity of  $w(o_i)$  for every object  $o_i$ . For weighted logic-based formulas, a sufficient condition for monotonicity is to require that the weight of every formula is positive, together with forbidding the negation symbol  $\neg$ .

Unless explicitly stated, we will consider in this chapter that all the preference relations we are dealing with are monotonic.

### 12.1.5 Multiagent Resource Allocation Settings

After this discussion about preferences, we can update the definition of MARA setting proposed at the end of the introduction and make it more precise. In the following, an *ordinal MultiAgent Resource Allocation setting* (ordinal-MARA setting for short) will be defined as a triple  $\langle N, \mathcal{O}, R \rangle$ , where  $N$  is a finite set of agents,  $\mathcal{O}$  is a finite set of indivisible and non-shareable objects, and  $R$  is a set  $\{\succsim_1, \dots, \succsim_n\}$  of preorders on  $2^{\mathcal{O}}$ , defined as well-formed formulas in a (compact) ordinal representation language.

A *cardinal-MARA setting* will be defined accordingly by replacing the set  $R$  of preorders by a set  $U = \{u_1, \dots, u_n\}$  of utility functions on  $2^{\mathcal{O}}$ , defined as well-formed formulas in a (compact) numerical representation language.

## 12.2 The Fairness vs. Efficiency Trade-off

Now that the setting is properly defined, we will deal with the definition of *fair* allocations. In what follows we will mainly focus on two notions of fairness (see Chapter 11, Thomson, 2015): *maxmin* allocations and *envy-free* allocations. Note that the former is only defined in the cardinal-MARA setting, because it requires the ability to compare the well-being of different agents, whereas the latter is well defined in both MARA settings.

In cardinal-MARA settings, maxmin allocations optimize the so-called *egalitarian social welfare*:

**Definition 12.10** (Maxmin) An allocation is maxmin when the utility of the poorest agent is as high as possible, *i.e.*

$$\max_{\pi \in \Pi} \left\{ \min_{i \in N} u_i(\pi(i)) \right\}$$

Note that it is still possible to conceive ordinal versions of this notion: for instance we may wish to maximize the worst “rank” of a bundle in the preference orderings of agents (*rank-maxmin*).

Envy-freeness only requires an ordinal-MARA setting to operate:

**Definition 12.11** (Envy-freeness) An allocation is envy-free when  $\pi(i) \succeq_i \pi(j)$  for all agents  $i, j \in N$ .

Unfortunately, these fairness objectives may not be compatible with the objective of *efficiency*. Informally, efficiency can be seen as the fact that resources shall not be “under-exploited”. At the weakest sense, it means that we should only consider *complete* allocations (objects should not be thrown away). However, usually, efficiency corresponds to the stronger notion of *Pareto-efficiency* or to the even stronger notion of *utilitarian optimality* (for cardinal-MARA settings). This latter notion of efficiency provides a convenient way to quantify the loss of efficiency due to the requirement to meet a fairness criterion: this is the idea of the *price of fairness*, which will also be discussed in Chapter 13 (Procaccia, 2015) in the context of divisible goods.

### 12.2.1 Maxmin allocations

As a warm-up, let us start with maxmin allocations and Pareto-efficiency. Observe that a maxmin allocation is not necessarily Pareto-optimal. This is so because this notion only focuses on the well-being of the agent who is worst-off, and overlooks the rest of the society. But it may well be the case that for the same utility enjoyed by the “poorest” agent, a better allocation of resources exists for the rest.<sup>10</sup> On the other hand, among the set of maxmin optimal allocations, one can easily see that at least one of them must be Pareto-optimal (many of them can be). Assume for contradiction that it is not the case. Then for each maxmin allocation  $\pi$  there is another allocation  $\pi'$  Pareto-dominating  $\pi$  and not in the set of maxmin allocations. Since  $\pi'$  Pareto-dominates  $\pi$  we have  $u_i(\pi'(i)) \geq u_i(\pi(i)) \geq \min_{k \in N}(\pi(k))$  for all  $i \in N$ . Hence either  $\min_{k \in N}(\pi'(k)) = \min_{k \in N}(\pi(k))$ , in which case  $\pi'$  is maxmin

<sup>10</sup> A way to overcome this problem and reconcile maxmin allocations with Pareto-optimality is to use the leximin preorder (Sen, 1970) which can be seen as a refinement of the maxmin fairness criterion: if two allocations yield the same maxmin value, then the leximin criterion will discriminate them based on the second poorest agent if possible, otherwise on the third poorest, and so on.

optimal, or  $\min_{k \in N}(\pi'(k)) > \min_{k \in N}(\pi(k))$ , in which case  $\pi$  is not maxmin optimal. In both case, this is a contradiction.

If we now consider the utilitarian notion of efficiency, then no guarantee can be given on the loss of efficiency induced by the requirement to have a maxmin allocation. Let us make this statement more formal using the notion of *price of fairness*. The price of fairness is usually defined as the ratio between the total utility of the optimal utilitarian allocation over the total utility of the best maxmin optimal allocation. The following holds:

**Theorem 12.12** (Caragiannis et al., 2012) *The price of fairness for maxmin allocations is unbounded.*

*Proof* Suppose the preferences of  $n$  agents regarding  $n$  objects are normalized so that they sum up to 1, and are set as follows: each agent  $i$  (from 1 to  $n - 1$ ) has utility  $\varepsilon$  for object  $o_i$ ,  $1 - \varepsilon$  for object  $o_{i+1}$ , and 0 for the other objects, while agent  $n$  has only utility 1 for object  $o_n$ . The maxmin allocation assigns object  $o_i$  to each agent  $i$ , and thus yields  $1 + (n - 1) \cdot \varepsilon$  overall when we sum utilities. But giving to each agent  $i \in 1, \dots, n - 1$  object  $o_{i+1}$  (and object 1 to anyone) yields an overall  $(n - 1) \cdot (1 - \varepsilon)$ . Hence, the ratio is unbounded as  $n$  grows.  $\square$

### 12.2.2 Envy-freeness

As an obvious first remark, note that a partial allocation where each good is thrown away is obviously envy-free: all agents own the same empty bundle, so they cannot envy each other. Thus, in this section we will focus on the non-trivial case of complete allocations.

First, as with the maxmin criterion, an envy-free allocation is not necessarily Pareto-efficient, as shown in the following example:

**Example 12.13** Let  $\mathcal{O} = \{o_1, o_2, o_3, o_4\}$  be a set of objects shared by two agents. Assume  $u_1(\mathcal{S}) = \mathbf{1}_{\{o_1, o_2\}}(\mathcal{S})$  and  $u_2(\mathcal{S}) = \mathbf{1}_{\{o_3, o_4\}}(\mathcal{S})$ , where  $\mathbf{1}$  is the indicator function. Then, the allocation where agent 1 owns  $\{o_1, o_3\}$  and agent 2 owns  $\{o_2, o_4\}$  is complete and envy-free, but not Pareto efficient: giving  $\{o_1, o_2\}$  and  $\{o_3, o_4\}$  respectively to agents 1 and 2 will strictly increase their utility function.

Next, it is easy to show that there does not always exist an envy-free complete allocation. Consider the case where two agents share a single good, and suppose this good is preferred by both agents to the empty bundle. Then, the agent owning the good will be envied by the other agent. More generally, the probability of existence of complete and envy-free allocation has actually been further investigated in a recent work by Dickerson et al. (2014). In particular, this work shows analytically that under several assumptions on the probability distribution of the agents' (additive) preferences, an envy-free allocation is unlikely to exist up to a given threshold on the ratio between the number of goods and the number of agents, and

very likely to exist beyond. Experimental results show an interesting phenomenon of phase-transition.

Finally, consider the utilitarian notion of efficiency. Similarly to the price of fairness, the price of envy-freeness has been defined by Caragiannis et al. (2012) as the ratio between the total utility of the optimal utilitarian allocation over the total utility of the best envy-free allocation. Caragiannis et al. (2012) show that the price of envy-freeness is  $\Theta(n)$ . So as more and more agents appear in the system, the gap between envy-free allocations and optimal allocations will grow at a linear rate.

### 12.2.3 Other fairness criteria

Beyond maxmin fairness and envy-freeness, *proportionality* is another prominent fairness criterion. This property, coined by Steinhaus (1948) in the context of continuous fair division (cake-cutting) problems, states that each agent should get from the allocation at least the  $n^{\text{th}}$  of the total utility she would have received if she were alone. Obviously, this criterion is related to maxmin fairness when utility are normalized (each agent gives the same value to the entire set of objects): if there exists an allocation that satisfies proportionality, then any maxmin-optimal allocation satisfies it. As we will see in Chapter 13 (Procaccia, 2015), it is always possible to find an allocation that satisfies proportionality. In the case of two agents, the example procedure given in the introduction of the chapter (Alice cuts, Bob chooses) obviously guarantees a proportional share to both agents. Once again, things turn bad when we switch to indivisible objects (just consider again one object and two agents, no allocation can give her fair share to each agent).

Even if it is not possible to guarantee the  $n^{\text{th}}$  of the resource to each agent, Demko and Hill (1988) have shown that under additive numerical preferences it is always possible to find an allocation guaranteeing a given amount of utility (only depending on the maximum weight  $\alpha$  given by the agents to the objects) to each agent. Markakis and Psomas (2011) have significantly extended this result, first by showing that it is actually possible to guarantee that the minimal amount of utility received by each agent  $i$  depends on the maximum weight  $\alpha_i$  given by this agent to the objects, and secondly by exhibiting a deterministic polynomial-time algorithm to compute it. The same idea has been used by Gourvès et al. (2013), who further refine these results by constructively exhibiting a stronger lower bound, that also works for fair division problems with a particular kind of admissibility constraints represented as a matroid.

Another approach has been proposed by Budish (2011). Instead on focusing on the maximal fraction of absolute utility it is possible to guarantee to each agent, Budish (2011) proposes to start from the “I cut you choose” protocol described earlier in the divisible case, and to adapt it to the indivisible case. According to this definition of fairness, every agent  $i$  should receive from the allocation at least what she would receive in the worst case if she had to partition the objects into  $n$

bundles and let the other  $n - 1$  agents choose first. In other words, each agent should receive at least the best (max), among all possible allocations (cuts), of the worst (min) share of this allocation: Budish (2011) calls it the *maximin share*. Obviously, in the cake-cutting case, this notion coincides with proportionality.

**Example 12.14** Consider a MARA setting involving two agents with additive preferences, and four objects  $\{o_1, o_2, o_3, o_4\}$ . Let agent 1's preferences be defined as follows:  $u_1(o_1) = 7$ ,  $u_1(o_2) = 2$ ,  $u_1(o_3) = 6$  and  $u_1(o_4) = 10$ . Then agent 1's maximin share is 12, associated to partition  $\{o_1o_3, o_2o_4\}$ .

Contrary to proportionality, in the case of additive preferences, maximin share guarantee is *almost* always possible to satisfy (Bouveret and Lemaître, 2014). Actually, Procaccia and Wang (2014) have exhibited some MARA-settings where no allocation guaranteeing maximin shares to everyone can be found, but these instances are rather rare.<sup>11</sup> Moreover, Bouveret and Lemaître (2014) notice that in the special case of additive preferences, not only envy-freeness implies proportionality, but also proportionality implies maximin share guarantee. It means that these properties form a scale of fairness criteria, from the strongest to the weakest.<sup>12</sup> This suggests another solution to the fairness vs. efficiency trade-off: try to satisfy envy-freeness if possible; if not, try to satisfy proportionality if you can; and finally, as a fallback fairness criterion, maximin share guarantee is almost always possible to satisfy.

### 12.3 Computing Fair Allocations

We will now see how challenging computing optimal fair allocations is. To achieve this, we will among other things study the computational complexity of decision problems associated with the computation of fair allocations. The input of these decision problems will include the preference profiles encoded in a given representation language. Note that if a preference profile is represented with a formula whose size is superpolynomial in  $p$  and  $n$ , then even if the decision problem is computationally easy, finding a fair allocation may remain prohibitive in practice —hence the relevance of compact representation languages discussed in the previous section.

#### 12.3.1 Maximin allocations

We start with a bad news: if we make no assumption on the preferences of agents (beyond monotonicity), then not only is computing an optimal maximin allocation computationally hard, but even computing an approximation is (Golovin, 2005). The argument is simple, and based as usual on a problem known to be hard. In the

<sup>11</sup> They also show that it is always possible to guarantee at least  $2/3$  of the maximin share to everyone.

<sup>12</sup> Actually there are two additional criteria in the scale, which we do not discuss here for the sake of clarity and conciseness.



PARTITION problem, we are given a collection of (positive) integers  $C = \langle c_1, \dots, c_q \rangle$  such that  $\sum_{i=1}^q c_i = 2k$ , and we are asked whether there exists  $I \subseteq \{1, \dots, n\}$  with  $\sum_{i \in I} c_i = k$ . But now take  $\mathcal{O} = C$  and set the utility functions of two agents as follows:

$$u(\mathcal{S}) = \begin{cases} 1 & \text{if } \sum_{x \in \mathcal{S}} x \geq k \\ 0 & \text{otherwise} \end{cases}$$

The only situation where an allocation with social welfare 1 can be obtained is when agents receive a bundle such that  $\sum_{x \in \mathcal{S}} x = k$ , otherwise any allocation yields utility 0 (because at least one agent enjoys utility 0). But then any approximation would have to distinguish between these cases, which requires to solve PARTITION. The careful reader should be skeptical at this point: shouldn't the complexity precisely depend on the size of the representation of  $u$ ? In fact, Golovin (2005) circumvents this problem of compact representation by assuming a “value query model” where an oracle can provide values of bundles of items in unit computation time. But even using the naive bundle form language, similar conclusions can be obtained, as long as allocations are required to be complete (Nguyen et al., 2013).

Getting more positive results requires further restrictions on the preferences of agents. However, even quite severe restrictions turn out to be insufficient. For instance, the problem remains inapproximable as soon as  $k \geq 3$ -additive functions are considered (Nguyen et al., 2013, 2014). Note that, as Nguyen et al. (2014) show, inapproximability results also hold for other “fair” collective utility functions, such as the Nash product for example.<sup>13</sup>

In fact, as we shall see now, even the most basic setting remains very challenging.

*The Santa Claus problem.* Take a cardinal-MARA setting, where utility functions are modular. This setting has been popularized as the *Santa Claus problem* (Bansal and Sviridenko, 2006): Santa Claus has  $p$  gifts to allocate to  $n$  children having modular preferences; Santa Claus allocates the gifts so as to maximize the utility of the unhappiest child (which is exactly the maxmin allocation). First, note that the problem remains NP-hard even in this restrictive setting (Bezáková and Dani, 2005; Bouveret et al., 2005). Furthermore, the problem cannot be approximated within a factor  $> 1/2$  (Bezáková and Dani, 2005).

There is a natural integer linear program (ILP) formulation for this problem, usually called the “assignment LP”. By taking  $\mathbf{x}_{i,j}$  to be the binary variable taking value 1 when agent  $i$  receives object  $o_j$ , and 0 otherwise, we set the objective

<sup>13</sup> The Nash product is defined as the product of all utilities.

function to be the maximization of the right-hand side of Inequality (12.6).

$$\text{maximize } \mathbf{y} \tag{12.3}$$

$$\forall i \in N, \forall j \in \mathcal{O} : \quad \mathbf{x}_{i,j} \in \{0, 1\} \tag{12.4}$$

$$\forall j \in \mathcal{O} : \quad \sum_{i \in N} \mathbf{x}_{i,j} = 1 \tag{12.5}$$

$$\forall i \in N : \sum_{o_j \in \mathcal{O}} w_i(o_j) \cdot \mathbf{x}_{i,j} \geq \mathbf{y} \tag{12.6}$$

The bad news is that just solving the relaxation of this ILP (that is, solving the problem by assuming that goods are divisible) is not a good approach since the *integrality gap* (the ratio between the fractional and the integral optimum) can be infinite. Indeed suppose there is a single object to allocate, for which every agent has the same utility, say  $x$ . Then the fractional solution would be  $x/n$ , while the ILP would yield 0 (Bezáková and Dani, 2005).

The similarity of the problem with scheduling problems is important to emphasize (take agents as being the machines, and objects as being the jobs). In particular the minimum makespan problem, which seeks to minimize the maximal load for an agent, is well studied. While the objective is opposite, this proved to be a fruitful connection, and motivated the use of (adapted) sophisticated rounding techniques (Lenstra et al., 1990). Bezáková and Dani (2005) were among the first to exploit this connection. They used job truncations techniques to propose an  $O(n)$  approximation, later improved to  $O(\frac{1}{\sqrt{n \log^3 n}})$  by Asadpour and Saberi (2010).

Linear programming is not the only possible approach to this problem: *branch and bound* techniques have also been investigated (Dall’Aglío and Mosca, 2007). For this type of algorithms, the quality of the bound is a crucial component. Dall’Aglío and Mosca (2007) use the “adjusted winner” procedure (that we discuss later on in this chapter) to compute this bound.

### 12.3.2 Computing envy-free or low-envy allocations

There is a simple algorithm which always returns an envy-free allocation: throw all the objects away! However, as discussed already, as soon as a very minimal efficiency requirement of *completeness* is introduced, an envy-free allocation may not exist. In fact, it is computationally hard to decide whether such an allocation exists (Lipton et al., 2004). If we now ask for an allocation which meets both envy-freeness and Pareto-optimality, then for most compact representation languages the problem lies above NP (Bouveret and Lang, 2008). More precisely, this problem is  $\Sigma_2^P$ -complete for most logic-based languages introduced in Section 12.1, including the very simple language leading to dichotomous preferences. It also turns out that the combinatorial nature of the domain plays little role here: even in additive domains, deciding whether there exists an efficient and envy-free allocation is  $\Sigma_2^P$ -complete (de Keijzer et al., 2009).

Given this, a perhaps more realistic objective is to seek to minimize the “degree

of envy” of the society. There are several ways to define such a metric. For example, Cavallo (2012) defines the *rate of envy* as the average envy of all agents. Here we follow Lipton et al. (2004) in their definitions:

$$e_{ij}(\pi) = \max\{0, u_i(\pi(j)) - u_i(\pi(i))\}$$

Now the envy of the allocation is taken to be the maximal envy among any pair of agents, *i.e.*:

$$e(\pi) = \max\{e_{ij}(\pi) \mid i, j \in N\} \quad (12.7)$$

*Allocations with bounded maximal envy.* One may ask whether allocations with bounded maximal envy can be found. The question is raised by Dall’Aglia and Hill (2003) and later addressed by Lipton et al. (2004). We will see that such bounds can be obtained, by taking as a parameter the *maximal marginal utility* of a problem, noted  $\alpha$ . The marginal utility of a good  $o_j$ , given an agent  $i$  and a bundle  $\mathcal{S}$ , is the amount of additional utility that this object yields when taken together with the bundle. Then the maximal marginal utility is simply the maximal value among all agents, bundles, and objects. In an additive setting, this is thus simply the highest utility that an agent assigns to a good.

The result by Lipton et al. (2004) —which improves upon a first bound of  $O(\alpha n^{3/2})$  given by Dall’Aglia and Hill (2003)— is then simply stated:

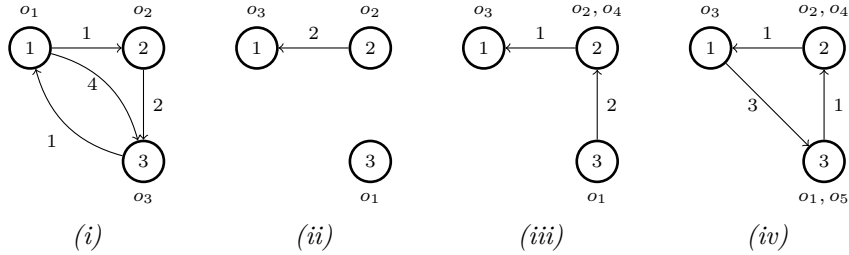
**Theorem 12.15** (Lipton et al., 2004) *It is always possible to find an allocation whose envy is bounded by  $\alpha$ , the maximal marginal utility of the problem.*

*Proof* (Sketch) First, we introduce the notion of the envy graph associated with an allocation  $\pi$ , where nodes are agents and there is an edge from  $i$  to  $j$  when  $i$  envies  $j$ . Now take a cycle in this envy graph: a key observation is that by rotating the bundles held by agents in the direction opposite to that of the cycle (so that each agent gets the bundle of the agent he envies), we necessarily “break the envy cycle” at some point. This is so because the utility of each agent in this cycle is increased at each step of this rotation. Furthermore, agents outside the cycle are unaffected by this reallocation. Now consider the following procedure. Goods are allocated one by one. First allocate one good arbitrarily. Now consider the end of round  $k$ , and suppose  $\{o_1, \dots, o_k\}$  have been allocated, and that envy is bounded by  $\alpha$ . At round  $k+1$  we build the envy graph. Next we rotate the bundles as previously described. As already observed, at some point there must be an agent  $i$  that no one envies. We then allocate object  $o_{k+1}$  to this agent  $i$ . Envy is thus at most  $\alpha$ .  $\square$

**Example 12.16** Let  $\mathcal{O} = \{o_1, o_2, o_3, o_4, o_5\}$  be a set of objects, and let  $\{1, 2, 3\}$  be three agents whose additive preferences are defined as follows:

$\mathcal{S}$	$o_1$	$o_2$	$o_3$	$o_4$	$o_5$
$u_1(\mathcal{S})$	1	2	5	3	7
$u_2(\mathcal{S})$	2	6	8	1	5
$u_3(\mathcal{S})$	5	4	4	3	1

The maximal marginal utility is 8. We know that by applying the procedure we are guaranteed to obtain an allocation with a degree of envy at most 8. Suppose we allocate the first three items  $o_i$  to agent  $i$ , we thus get  $\pi(1) = \{o_1\}$ ,  $\pi(2) = \{o_2\}$ , and  $\pi(3) = \{o_3\}$ . At this step of the procedure, the envy graph is depicted in (i). For convenience we indicate the degree of envy on each arrow. There are two cycles. Let us consider for instance the cycle (1,3), and rotate (this corresponds to simply swapping the resources of agents 1 and 3 here). It happens to be sufficient to remove the cycle: we obtain the new envy graph (ii). Now we wish to allocate  $o_4$ . No one envies 2 nor 3, so we can for instance allocate  $o_4$  to 2, resulting in (iii). The graph is without cycle. We can now give  $o_5$  to agent 3, thus obtaining (iv), with a degree of envy of 3.



Note that in that case, by performing a final rotation, we could obtain an envy-free allocation.

Observe that the procedure makes no assumption on the preferences of agents. The result thus shows that it is always possible to have an allocation bounded by the highest marginal value. Of course, such a bound is tight, as is easily observed by a scenario involving a single good and two agents with the same utility for it. In general however, for a given instance, allocations with a much lower envy than this bound will exist.

*Low-envy allocations.* Is it possible to design algorithms returning an allocation with minimal envy, or at least an approximation of it? A critical problem is that in the case of general preferences, the amount of information that needs to be transmitted to the algorithm is prohibitive (see Section 12.4). It is thus natural to consider the same question in restricting the domain considered.

Another technical issue occurs: the minimum degree of envy as defined by Equation (12.7) is 0 when the allocation is envy-free. While this an intuitive requirement,

it has an undesirable consequence: any finite approximation would have to be able to distinguish an envy-free allocation. Unfortunately, as we have seen, this problem is hard, even in the case of modular preferences. Thus, unless  $P = NP$ , there is no hope for approximation here (remember the same line of argumentation was used to show inapproximability for maxmin allocations).

To circumvent this, a different measure of envy is considered. The *minimum envy-ratio* is defined as:

$$\max \left\{ 1, \frac{u_i(\pi_j)}{u_i(\pi_i)} \right\}$$

When the objective function is to minimize this measure, positive results can be obtained. Lipton et al. (2004) were the first to address this version of the problem. They made the additional assumption that agents have the same preferences. In this context, the greedy procedure proposed by Graham (1969) in the context of scheduling yields a 1.4-approximation. The procedure is fairly simple: rank the goods in decreasing order, then allocate them one by one, to the agent whose current bundle has least value. But Lipton et al. (2004) went further: they showed that it is possible to achieve an approximation arbitrarily close to 1 with an algorithm running in polynomial time in the input size (in other words, a Polynomial-Time Approximation Scheme). When the number of agents is bounded, it is even possible to get an FPTAS for this minimization problem. Recently, Nguyen and Rothe (2013) took up this research agenda. When the number of agents is bounded, they obtain an FPTAS for this minimization problem (and other degree measures), *even when agents have different preferences*. On the other hand, they showed that when the number of agents is part of the input, it will not be possible to obtain (in polynomial time) an approximation factor better than  $3/2$ , under the usual  $P \neq NP$  assumption.

*Envy-freeness and ordinal preferences.* Let us conclude this overview of computational aspects of envy-freeness with a quick look at ordinal preferences. An interesting feature of envy-freeness is that this notion does not require any interpersonal comparison of preferences. As a result, envy-freeness is a *purely ordinal* notion: this fairness criterion is properly defined as soon as the agents are able to compare pairs of bundles (which is not the case for maxmin fairness, requiring cardinal preferences). The ordinal analogous of the problem studied by Lipton et al. (2004) for numerical preferences, namely, the problem of finding an efficient and envy-free allocation with ordinal preferences (using pairwise dominance for lifting preferences from objects to bundles) has been studied by Brams et al. (2004); Brams and King (2005) and later by Bouveret et al. (2010); Aziz et al. (2014).

The main difficulty here is that, unlike additivity for numerical preferences, only requiring responsiveness (and monotonicity, for the two latter references) leaves many pairs of bundles incomparable. For example, if we have  $o_1 \succ o_2 \succ o_3 \succ o_4$ , responsiveness implies that  $o_1 o_2 \succ o_3 o_4$ , but leaves  $o_1 o_4$  and  $o_2 o_3$  incomparable.

This calls for an extended version of envy-freeness that takes into account incomparabilities. Brams et al. (2004); Bouveret et al. (2010) propose the two notions of *possible* and *necessary envy*:<sup>14</sup> basically, agent  $i$  possibly (resp. necessarily) envies agent  $j$  if  $\pi(i) \not\prec_i \pi(j)$  (resp.  $\pi(j) \succ_i \pi(i)$ ). The recent work of Aziz et al. (2014) further extends and refines these notions by introducing a new definition of ordinal dominance based on stochastic dominance.

On the positive side, it turns out that the problem of determining whether a possible envy-free efficient allocation exists is in P (Bouveret et al., 2010) for strict preferences, for different notions of efficiency (completeness, possible and necessary Pareto-efficiency). On the negative side, things seem to be harder (NP-complete) for necessary envy-freeness, and as soon as ties are allowed (Aziz et al., 2014). Note also that in the case of ordinal preferences, defining measures of envy makes less sense than for numerical preferences. That means that it seems difficult to use approximation to circumvent the computational complexity of the problem.

### 12.3.3 Other fairness measures

In this section we have looked in details at the computation of maxmin or low-envy allocations. Of course, there are many other criteria of fairness of interest. In particular, it is natural to not only focus on the worst-off agent, but to define a more general measure of the inequality of the society and to rely on a *generalized Gini social-evaluation function*. This class of functions is also known as *ordered weighted averages* (Yager, 1988). The computation of these functions has been studied by Lesca and Perny (2010). They investigate in particular how techniques of linear programming (such as the one mentioned in Section 12.3.1) can be adapted so to handle these problems. Recently, the problem of computing inequality indices in combinatorial domains has been considered by Endriss (2013). Also, Vetschera (2010) came up with an approach generalizing the branch and bound approach of Dall’Aglia and Mosca (2007) to a wider class of objective functions: more precisely, for any setting involving the division of indivisible goods between *two* agents, and for which the objective function is maximum when the utilities of both players are equal (in the hypothetical continuous case). In that case, the bound based on the “adjusted winner” split, which will be presented in section 12.4.1, is not valid any longer.

## 12.4 Protocols for Fair Allocation

The *centralized* “one-shot” approaches to fair resource allocation we have considered so far work in two steps: first the agents fully reveal their preferences (may them be ordinal or numerical) to the benevolent arbitrator, then this arbitrator computes a

<sup>14</sup> Actually Brams et al. (2004) use a different terminology, but the idea is the same.

satisfactory allocation (thanks to an algorithm) and gives the objects to the agents. This approach has two main drawbacks: (i) the elicitation process can be very expensive or agents may not be willing to fully reveal their preferences; and (ii) agents may be reluctant to accept a solution computed as a black-box.

Regarding (i), there is not much we can do in the worst case : when preferences are not modular, the communication load that is required to compute optimal (or indeed approximated) fair solutions becomes a fundamental barrier. This can be stated more formally:

**Theorem 12.17** (Lipton et al., 2004; Golovin, 2005) *Any deterministic algorithm would require an exponential number of queries to compute any finite approximation for the minimal envy problem (Lipton et al., 2004), or maxmin allocation (Golovin, 2005).*

Such lower bounds can be obtained by borrowing techniques from the communication complexity literature (see also Chapter 10, Boutilier and Rosenschein, 2015).

The incremental protocols we discuss in this section take a different approach: they prescribe “simple” actions to be taken by the agents at different stages of the process (comparing two bundles, choosing an item, etc.), and they (typically) do not require heavy computation from the central authority. They sometimes do not involve any central computation at all (beyond verification of the legality of agents’ actions) nor preference elicitation, and may even work in the absence of a central authority in some cases, as we shall see.

Before we go on and present original protocols, note that some of the algorithms presented earlier in this chapter can be readily interpreted as protocols. This is true in particular of the procedure of Lipton et al. (2004): allocate items sequentially, and each time a new item is assigned, ask agents to point to agents they envy (note that this does not require to elicit preferences). When a cycle occurs, rotate the bundles as indicated in the procedure, and ask again agents who they envy, etc. Of course, as we have already seen, the guarantees of such a protocol are not so good. We will see that better guarantees can be given, at the price of restrictions on the number of agents, or on the type of preferences.

#### 12.4.1 Protocols for two agents

*The adjusted winner procedure.* This procedure has been used in various contexts (Brams and Taylor, 2000). It works for two agents, with additive utility functions. At the end of the procedure, one item may need to get split, but as we don’t know beforehand which one, it has to assume that all items are divisible. The technique is nevertheless inspiring, and can be used to compute a bound in the indivisible case (Dall’Aglio and Mosca, 2007), as already mentioned.

In the first phase of the algorithm (the “winning phase”), goods are allocated efficiently, that is, each good is assigned to the agent who values it the most. At

the end of this stage, either  $u_1 = u_2$  and we are done, or some agent (say  $r$ , the richest) has a higher utility than the other (say  $p$ , the poorest) and the “adjusting phase” can begin. During this phase, goods are transferred from the richest to the poorest, in increasing order of the ratio  $\frac{u_r(o)}{u_p(o)}$  (note that the ratio is necessarily  $\geq 1$ ). The algorithm stops when either both agents enjoy the same utility, or the richest becomes the poorest. Suppose this happens under the transfer of good  $g$ : then  $g$  is split so as to attain *equitability*, i.e. the utility of both agents is equal. To get this equitable outcome, the richest gets a fraction of  $g$  computed as follows:

$$\frac{u_p(g) + u_p(\pi(p) \setminus \{g\}) - u_r(\pi(r) \setminus \{g\})}{u_r(g) + u_p(g)}$$

The allocation is thus also a maxmin allocation. In fact, it has several desirable properties:

**Theorem 12.18** (Brams and Taylor, 2000) *The adjusted winner procedure returns an equitable, envy-free, and Pareto-optimal allocation.*

Recall that here “equitable” means that the two agents enjoy the same utility at the end of the protocol.

**Example 12.19** Let  $\mathcal{O} = \{o_1, o_2, o_3, o_4, o_5\}$  be a set of objects.

$\mathcal{S}$	$o_1$	$o_2$	$o_3$	$o_4$	$o_5$
$u_1(\mathcal{S})$	1	2	5	3	7
$u_2(\mathcal{S})$	2	6	8	1	5

After the winning phase, agent 2 gets  $\{o_1, o_2, o_3\}$  and agent 1 gets  $\{o_4, o_5\}$ . The utility of agent 2 is  $2+6+8=16$  while the utility of agent 1 is 10. Agent 2 will transfer goods to agent 1, starting with  $o_3$  (with ratio  $\frac{8}{5}$ , while  $o_1$  has  $\frac{2}{1}$ , and  $o_2$  has  $\frac{6}{2}$ ). But once we do that agent 1 becomes the richest: the good  $o_3$  has to be split, with agent 2 obtaining  $(5+10-8)/13 = 7/13$  of the good  $g$  (and the rest for agent 1). This provides each agent a utility  $\simeq 12.3$ .

*The undercut procedure.* Unlike the adjusted winner, this protocol takes as input ordinal information (a ranking of the items), and assumes that preferences are responsive (in fact, Aziz, 2014 has recently introduced a modified version of the procedure which works for the more general class of *separable* preferences).

As discussed in Section 12.1, the assumption of responsiveness allows to rank some of the bundles only: for instance if  $o_1 \triangleright o_2 \triangleright o_3$ , we know among others that  $o_1 o_2 \succ o_1 o_3 \succ o_1$ . The undercut procedure guarantees to find an envy-free allocation among two agents, whenever one exists. The procedure runs in two phases: in the *generation phase*, agents name their preferred item. If the items are different, they are allocated to agents asking them, otherwise they are placed in the



*contested pile*. This is iterated until all the items are either allocated or placed in the pile. Observe that at the end of the generation phase, each agent holds a bundle that she values more than the bundle held by the other agent. The main role of the protocol is then to implement a split of the “contested” items that will lead to an envy-free allocation. The key step is to let agents reveal what is called their *minimal bundles* (they may have several of them): a minimal bundle for agent  $x$  is a set of items that is worth at least 50% of the value of the full set of items for  $x$ —we say that such a bundle is envy-free (EF) to agent  $x$ —and such that it is not possible to find another bundle ordinally less preferred to it, which would also be EF to  $x$ . So for instance, with  $o_1 \triangleright o_2 \triangleright o_3$ , if  $o_1 o_3$  is EF to agent 1, then  $o_1 o_2$  cannot be a minimal bundle. Once minimal bundles have been named, the protocol chooses randomly one of them as a proposal (say  $\mathcal{S}_1$ , that of agent 1). Next agent 2 has the opportunity to either accept the complement of the proposal, or to undercut the proposal, by modifying the proposed split and take for herself a bundle strictly less preferred than  $\mathcal{S}_1$ .

**Theorem 12.20** (Brams et al., 2012) *If agents differ on at least one minimal bundle, then an envy-free allocation exists and the undercut protocol returns it.*

**Example 12.21** We borrow an example from Brams et al. (2012), where both agents declare the same ranking of five items  $o_1 \triangleright o_2 \triangleright o_3 \triangleright o_4 \triangleright o_5$ . In that case an envy-free split looks unlikely because agents have exactly the same preference: thus, after the generation phase, *all* items go to the contested pile. Now assume agent 1 announces  $o_1 o_2$  as her only minimal bundle, while agent 2 announces  $o_2 o_3 o_4 o_5$ . The minimal bundles differ: there must be an envy-free allocation. Let us see why. Suppose  $o_1 o_2$ , the minimal bundle of agent 1, is chosen for proposal. Then agent 2 will reject this proposal because  $o_3 o_4 o_5$  is not EF to her (as she declared  $o_2 o_3 o_4 o_5$  as minimal). It means that  $o_1 o_2$  must be EF to agent 2. But as  $o_1 o_2$  is not minimal to her, she may propose to take  $o_1 o_3$  which is the next ordinally less preferred bundle, and so must still be EF to her. So agent 2 may propose this split, letting agent 1 with  $o_2 o_4 o_5$ . As on the other hand,  $o_1 o_2$  was minimal to agent 1, it must be that  $o_1 o_3$  is worth less than 50%, and so  $o_2 o_4 o_5$  is EF to agent 1. This allocation is envy-free.

Compared to the adjusted winner, this protocol has the advantage to only require ordinal preferential information. But note that it may not be able to produce a complete envy-free allocation (at least in a deterministic way) Suppose we run the protocol on Example 12.19. Agent 1 reports  $o_5 \triangleright o_3 \triangleright o_4 \triangleright o_2 \triangleright o_1$ , while agent 2 reports  $o_3 \triangleright o_2 \triangleright o_5 \triangleright o_1 \triangleright o_4$ . Thus, after the generation phase, agent 1 gets  $o_5 o_4$  and agent 2 gets  $o_3 o_2$ : item 1 is the only contested item and thus no complete envy-free allocation of the pile can be proposed (but assigning this last item randomly may still yield an envy-free allocation).

### 12.4.2 Protocols for more than two agents

*Picking sequences.* Can we take inspiration from the generation phase of the under-cut procedure and allocate goods incrementally? This is soon going to be unpractical as the number of agents grows, since all goods will be likely to be contested. But an alternative solution is to fix beforehand a sequence among agents. This is viable even for a large number of agents, and only requires a *partial* elicitation of the agents' preferences (or, at the extreme, no elicitation at all). From the point of view of agents, the assumption of responsiveness of preferences suffices to decide simply which item to pick.

More precisely, the benevolent arbitrator defines a sequence of  $p$  agents. Every time an agent is designated, she picks one object out of those that remain. For instance, if  $n = 3$  and  $p = 5$ , the sequence 12332 means that agent 1 picks an object first; then 2 picks an object; then 3 picks two objects; and 2 takes the last object. Such a protocol has very appealing properties: first, it is very simple to implement and to explain<sup>15</sup> and secondly, it frees the central authority from the burden of eliciting the agents' preferences. Seen from the point of view of communication complexity, implementing such a protocol just requires the exchange of  $O(m \log(m))$  bits of information (at each of the  $m$  steps of the protocol, the chosen agent just needs to send the identifier of the object she wants to pick, which requires  $O(\log(m))$  bits). A classical centralized protocol as the ones we discussed earlier would require  $\Theta(nm \log(m))$  bits of information to send the agents' preferences to the arbitrator (if they only provide an ordinal information), and  $\Theta(m \log(n))$  additional bits for the arbitrator to send the result to the agents.

This protocol has been discussed to some extent by Brams and Taylor (2000), who focus on two particular sequences, namely *strict alternation*, where two agents pick objects in alternation, and *balanced alternation* (for two agents) consisting of sequences of the form 1221, 12212112, etc. One can feel intuitively that these kinds of sequences are quite fair, in the sense that alternating the agents in the sequence increases the probability of obtaining a fair allocation in the end (for example, the sequence 1221 is more likely to make both agents happy than 1122, where agent 2 is very likely to be disappointed). The problem of finding the best (fairest) sequence has been investigated by Bouveret and Lang (2011), who proposed a formalization of this problem based on the following hypotheses: (i) the agents have additive utilities; (ii) a scoring function maps the rank of an object in a preference relation to its utility value —the agents may have different rankings, but this scoring function is the same for all agents; (iii) the arbitrator does not know the agents' preferences but only has a probability distribution on the possible profiles. In this framework, the best sequence is just one that maximizes the expected (utilitarian or egalitarian) collective utility. Even if the precise complexity of the problem of finding the optimal sequence is still unknown, Kalinowski et al. (2013) have shown, among other things,

<sup>15</sup> The less understandable an allocation protocol is, the less likely it will be accepted by the agents.

that the strict alternation policy is optimal with respect to the utilitarian social welfare, if we consider two agents that can have any preference profile with equal probability, and a Borda scoring function. This formally proves the intuitive idea that under mild assumptions, a sequence like 12121212... maximizes the overall utility of the society.

*The descending demand procedure.* In this protocol proposed by Herreiner and Puppe (2002), agents are assumed to have a linear ordering over all subsets of resources (satisfying monotonicity). An ordering of the agents is fixed beforehand: one by one, they name their preferred bundle, then their next preferred bundle, and so on. The procedure stops as soon as a feasible complete allocation can be obtained, by combining only bundles mentioned so far in the procedure. There may be several such allocations, in which case the Pareto-optimal ones are selected. It does not offer any guarantee of envy-freeness, but produces “balanced” allocations, that is, allocations which maximize the rank in the preference ordering<sup>16</sup> of the bundle obtained by the worst-off agent. As mentioned already, this notion is the natural counterpart of the egalitarian social welfare in this specific case where linear orders are available.

**Theorem 12.22** (Herreiner and Puppe, 2002) *The descending demand procedure returns a Pareto-efficient and rank-maxmin-optimal allocation.*

This protocol is simple and can be used by more than two agents, for a moderate number of goods though (since otherwise the requirement to rank all subsets becomes unrealistic).

*Distributed fair division.* When many agents are involved in an allocation, fully distributed approaches can be well adapted. The main idea is that agents will start from an initial allocation, and myopically contract local exchanges (or deals) independently from the rest of the society.<sup>17</sup> In particular, this means that agents can rely on a *local rationality criteria* which tells them whether to accept or not a deal. A rationality criteria is *local* when it can be checked by inspecting only those agents who modified their bundle during the deal. Ideally, such deals should be “simple” (for instance, involving only two agents). For instance, based on the *Pigou-Dalton principle* (Moulin, 1988), we may conceive a system where only bilateral deals which diminish the inequality among agents involved are allowed.

The question is whether this type of incremental deal-based protocol has any chance in the end to converge to an optimal (fair) solution. As discussed by Endriss et al. (2006), the question is related to the *separability* (Moulin, 1988, page 43) of the social welfare ordering considered (not to be confused with the separability of

<sup>16</sup> Assuming lower ranks correspond to less preferred bundles.

<sup>17</sup> This is similar to the case of housing markets vs. house allocation problems, see Chapter 14 (Klaus et al., 2015).

individual preferences). The idea behind separability is the following: suppose that only a subset of agents are involved in a deal  $\delta$ . Then, the change in social welfare caused by  $\delta$  will always be the same, irrespective of the level of utility of the agents not concerned by the deal. To grasp the intuition, compare the utilitarian social welfare (separable), with maxmin (only separable in the weak sense) and the notion of envy (not separable). Suppose some agents implement a deal (while the rest of the agents don't), and that you can observe its outcome. If the sum of utility among those agents involved in the deal has increased, you know that the overall sum of utility must have increased as well. But you can never be sure that the min utility of the society has increased, even though you observe that the min utility among the agents involved has (because the agent who is currently the worst-off may not be involved in the deal). Still, the allocation cannot become worse (hence separability in the weak sense). This is not even the case with envy: the implementation of a local deal can have negative consequences, since by modifying agents' bundle the envy of agents outside the deal (but observing it) can certainly increase. In a series of paper by Sandholm (1998); Endriss et al. (2006); Chevaleyre et al. (2010), convergence results are proven for different social welfare measures, domain restrictions, and deal types. These results typically show convergence of *any* sequence of deals to some allocation where no further deal is possible, with guarantees on the quality of such a final allocation.

For instance, convergence to maxmin allocations by means of locally egalitarian deals (that is, deals where the situation of the worst-off agent involved has improved) can still be guaranteed by exploiting the separability of the (stronger) leximin social welfare. Of course the complexity of the problem has not magically disappeared. This is witnessed by two types of “negative” results, affecting the complexity of a single step (*i.e.* a deal), and the complexity of the sequence of deals as a whole:

- *any* kind of restriction on deal types ruins the guarantee of convergence in the general domain (Endriss et al., 2006). This is problematic since, as mentioned, deals are likely to be simple in practice (for instance, swapping two resources).
- the upper bound on the length of the sequence of deals can be exponential in the worst case (Sandholm, 1998; Endriss and Maudet, 2005), even when considering only the simplest type of deals, consisting of moving a single resource from one agent to another agent (Dunne, 2005).

On the positive side, these approaches can be deployed in the absence of a central authority, and they enjoy a nice anytime behavior: they return a solution even if stopped before convergence, and the quality of the obtained allocation usually improves as long as the agents can perform deals (though this may not be theoretically guaranteed for all social welfare measures, as briefly discussed above).

## 12.5 Conclusion

In this chapter we have discussed fair division problems involving indivisible items. We have seen that this setting poses several challenges, starting from the mere representation of agents' preferences, to the computation of optimal fair allocations (with maxmin and envy as main illustrations). These difficulties are not necessarily intertwined: we have seen for instance that even with additive preferences, the algorithmic challenge may remain surprisingly high. Of course the usual warning is flashing here: these are typically worst-case results, and recent work suggests that under specific assumptions about the domain considered, it may be possible to obtain satisfying allocations with high probability (and even compute them rather easily). For the design of practical interactive protocols, the preference representation and communication bottleneck seems more stringent, and indeed most efforts have concentrated so far on the setting of two agents equipped with additive, or at least responsive preferences. It is striking though, that very few works have addressed natural preference restrictions beyond such domains. An important question is how such protocols and algorithms will be adopted in practice, for instance whether agents may manipulate, and whether suggested solutions can be easily understood and accepted. While the allocation settings discussed here remain as general as possible, specific features may require dedicated approaches. For instance, agents may have different priority, they may enter the system sequentially, etc. These aspects (among many others of course) have been investigated in the matching community, and this leads us to strongly encourage the reader interested in fair division to jump to Chapter 14 (Klaus et al., 2015). Indeed, in particular when one resource exactly has to be allocated to each agent, allocation problems can be readily captured in a matching setting (where stability is the primary focus of interest). But if agents have priority when selecting their resource, the notion of envy may only be *justified* when an agent has higher priority over the agent he envies. Interestingly, this corresponds to the notion of stability. This illustrates how the concepts discussed in both chapters can be connected.

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