Efficiency, Sequenceability and Deal-Optimality in Fair Division of Indivisible Goods

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Picking sequences

➤ **Agents, objects, preferences**

- a set $\mathcal{N} = \{1, ..., n\}$ of agents
- a set $\mathcal{O} = \{o_1, ..., o_m\}$ of **indivisible** objects
- additive utility functions to represent preferences over resources

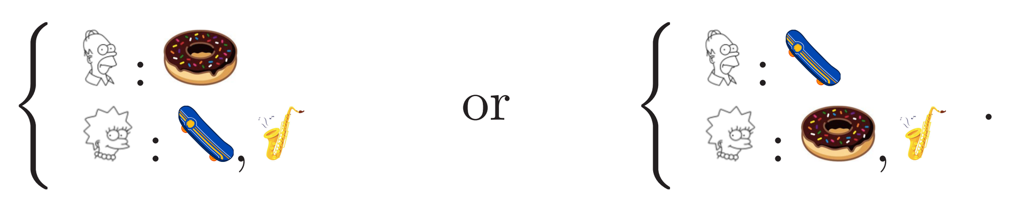
➤ **Picking sequences**

- A central entity chooses a sequence $\sigma \in \mathcal{N}^m$ of agents having as many agents as the number of objects.
- Then, each agent appearing in the sequence is asked to choose in turn one object among the remaining ones.
- Agents are assumed to make sincere choices (each agent chooses her preferred object among the remaining ones).

➤ **Example** Let us consider the following preferences:

	8	2	1
	5	1	5

The sequence ⟨ , , ⟩ can generate two distinct allocations:



➤ **Issues**

- Can we say something about the **efficiency** and the **fairness of the allocations obtained by picking sequences?**
- **How does the picking sequence relate to other protocols?**

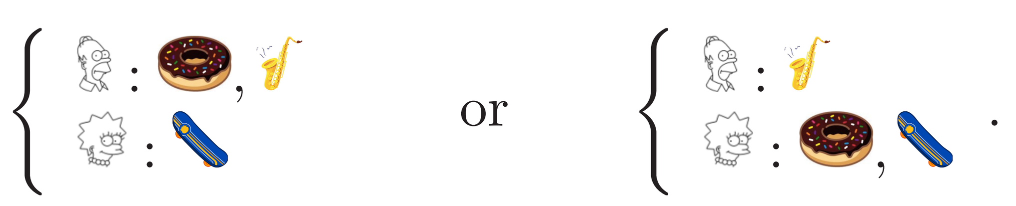
Sequenceable allocations

➤ An allocation is said **sequenceable** if there exists a sequence of sincere choices that generates it.

➤ Some allocations can be generated by several sequences:

is generated by 3 sequences.

But some allocations are not sequenceable:



Pareto-optimality

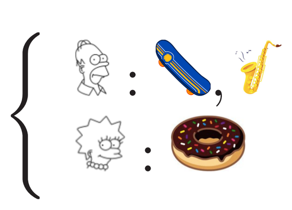
A sequence of sincere choices does not necessarily generate a Pareto optimal allocation **but**

☞ *Every Pareto-optimal allocation is sequenceable.*

	5	4	2
	8	2	1

The sequence ⟨ , , ⟩ generates the allocation:

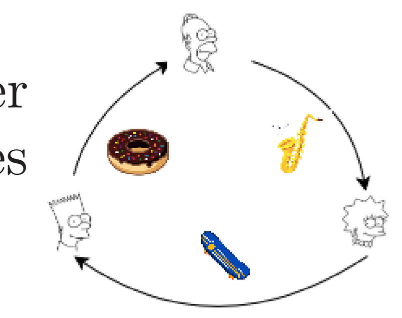
which is dominated by the allocation:



generated by ⟨ , , ⟩.

Cycle deals optimality

- In a **cycle deal**, each agent gives a subset of her items to the next agent in the cycle and receives some items from the previous agent.



- We consider **(weakly) improving cycles deals**, i.e. cycle deals where each agent (weakly) improves her utility.
- An allocation is said to be $>$-(N, M)-cycle optimal (resp. $\geq$-(N, M)-cycle optimal) if it does not admit any strictly (resp. weakly) improving (K, M)-cycle deal for any number of agents K with $K \leq N$ and number of objects M.

☞ *An allocation is sequenceable if and only if it is $>$-n-cycle optimal (with $n = |\mathcal{N}|$).*

➤ **Hierarchy of notions**:

P.O. → $\geq$-n-cycle opti ; P.O. → $>$-n-cycle opti

$\geq$-n-cycle opti ⟶ $>$-n-cycle opti ⟷ Sequenceable

$\geq$-n-cycle opti ↓ $\geq$-$n-1$-cycle opti ; $>$-n-cycle opti ↓ $>$-$n-1$-cycle opti

$\geq$-$n-1$-cycle opti ⟶ $>$-$n-1$-cycle opti

$\geq$-$n-1$-cycle opti ⇣ $\geq$-swap-cycle opti ; $>$-$n-1$-cycle opti ⇣ $>$-swap-cycle opti

$\geq$-swap-cycle opti ⟶ $>$-swap-cycle opti

Fairness of picking sequences

➤ **Envy-freeness**: an allocation is said to be envy-free when no agent strictly prefers the share of another agent to her own share.

☞ *There exist envy-free allocations that are not sequenceable.*

The circled allocation is envy-free and non-sequenceable.

	(12)	15	11	(7)	2
	2	12	(7)	15	(11)
	15	(20)	9	2	1

➤ **Competitive equilibrium for equal incomes (CEEI)**: an allocation $\vec{\pi} = \langle \pi_0, \ldots, \pi_n \rangle$ and a vector of prices $\vec{p} \in [0, 1]^m$ are said to form a *competitive equilibrium from equal incomes* if

$$\forall i \in \mathcal{N} : \pi_i \in \arg\max_{\pi \subseteq \mathcal{O}} \left\{ u_i(\pi) : \sum_{o_k \in \pi} p_k \leq 1 \right\}.$$

In other words, π_i is one of the maximal shares that i can buy with a budget of 1, given that the price of each object o_k is p_k.

It is already known that every CEEI is envy-free.

☞ *Every CEEI allocation is sequenceable.*

To go further

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