Efficiency, Sequenceability and Deal-Optimality in Fair Division of Indivisible Goods

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Fair division of indivisible goods...

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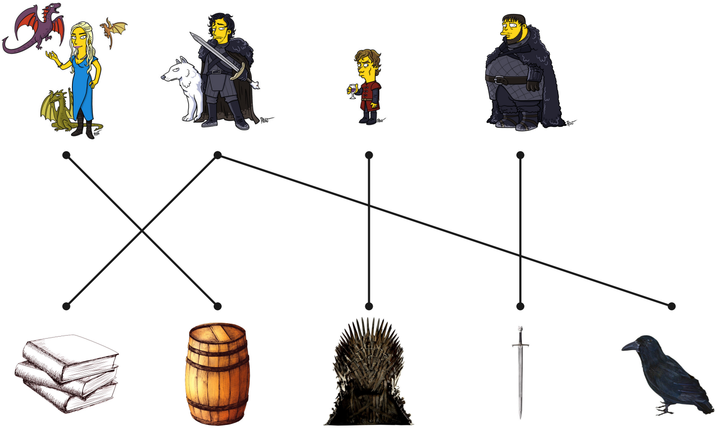

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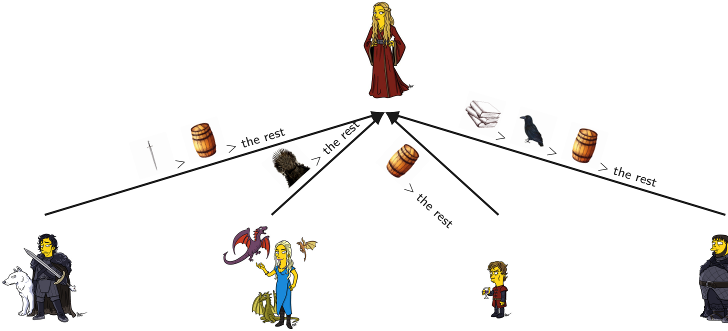

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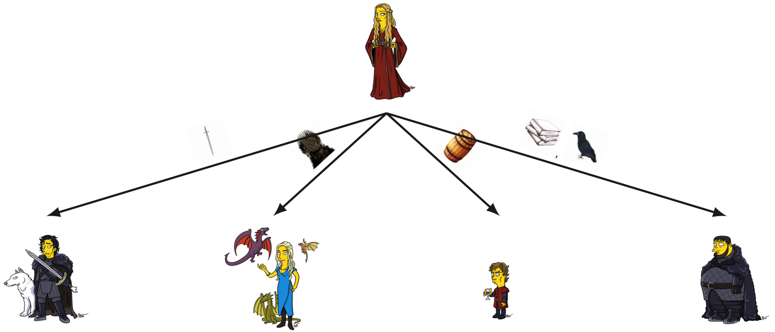

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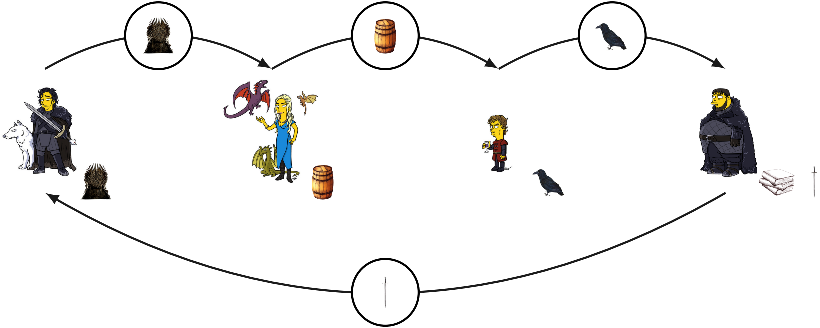

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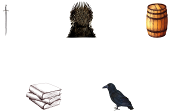

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In this work, we try to reconcile these approaches.

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$u_2(\{2, 3\}) = 1 + 6 = 7$

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- a complete allocation $\overrightarrow{\pi} : \mathcal{A} \rightarrow 2^{\mathcal{O}}$...
- ...which takes into account the agents' preferences.

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Part I

Sequences of sincere choices (aka picking sequences)

Sequences of sincere choices

A simple protocol:

1. fix a sequence of agents σ
2. ask the agents to pick in turn their preferred object

Studied a lot. See among others:

Bouveret, S. and Lang, J. (2011).
A general elicitation-free protocol for allocating indivisible goods.
In Walsh, T., editor, *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI-11)*, pages 73–78, Barcelona, Spain. IJCAI/AAAI.

Brams, S. J. and Taylor, A. D. (2000).
The Win-win Solution. Guaranteeing Fair Shares to Everybody.
W. W. Norton & Company.

Kohler, D. A. and Chandrasekaran, R. (1971).
A class of sequential games.
Operations Research, 19(2):270–277.

Allocations and sequences

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- **Step 2 :** agent 2 chooses o_3
- **Step 3 :** agent 2 chooses o_2

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Sequence $\sigma = 1, 2, 2$

- **Step 1 :** agent 1 chooses o_1
- **Step 2 :** agent 2 chooses o_3
- **Step 3 :** agent 2 chooses o_2

Final allocation: $\overrightarrow{\pi} = \langle \{1\}, \{2, 3\} \rangle$

Properties of sequential allocations

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First result: a precise characterization of sequenceable allocations.
We can decide in time $O(N \times M^2)$ if an allocation is sequenceable.

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agent 1	4	2	5
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However...

Second result: Every Pareto-efficient allocation is sequenceable.
But Pareto-efficiency $\not\Leftarrow$ sequenceability.

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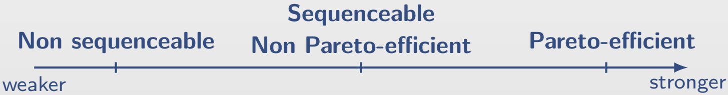

Part II

Negotiation...

Trading as an allocation procedure

Another allocation procedure.

- Start from an initial allocation
- Let the agents **trade** objects

A particular kind of trading scheme: **trading cycles**

Sandholm, T. W. (1998).
Contract types for satisficing task allocation: I. theoretical results.
In Sen, S., editor, *Proceedings of the AAAI Spring Symposium: Satisficing Models*, pages 68–75, Menlo Park, California. AAAI Press.

Shapley, L. and Scarf, H. (1974).
On cores and indivisibility.
Journal of mathematical economics, 1(1):23–37.

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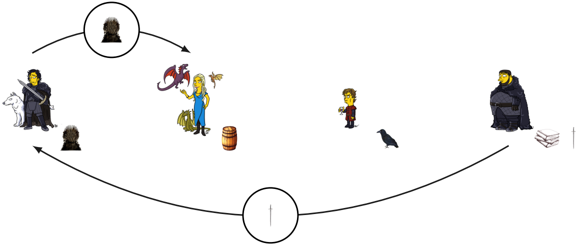

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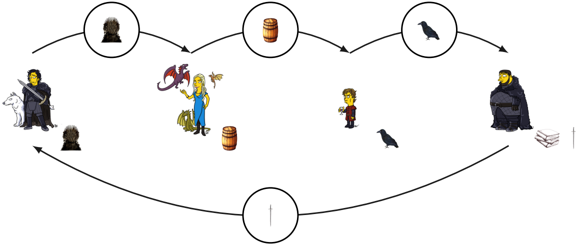

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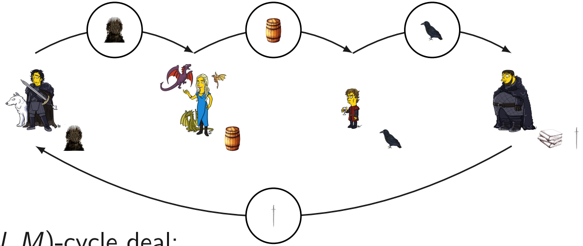

(N, M)-cycle deal:

- N: cycle length
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(in the example above, $N = 4$ and $M = 1$)

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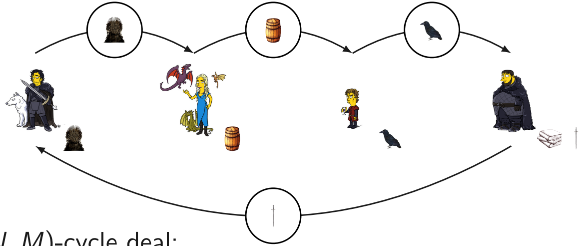

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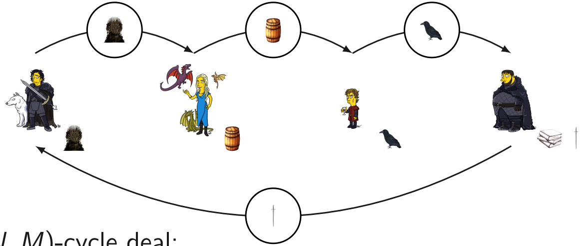

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Notion of efficiency: **cycle-deal optimality**.

Deals and efficiency

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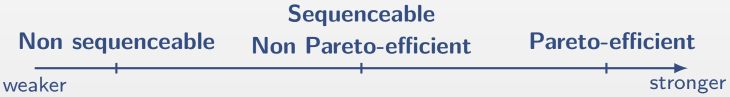

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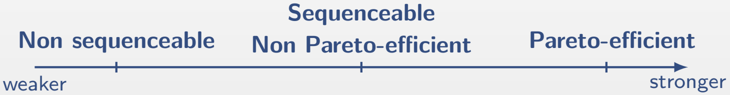

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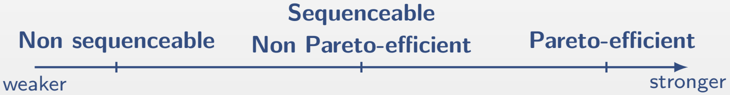

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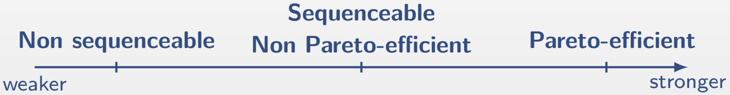

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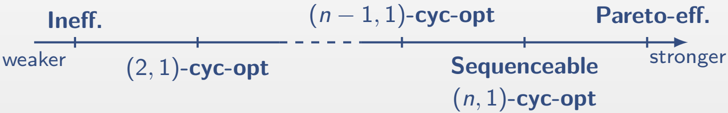

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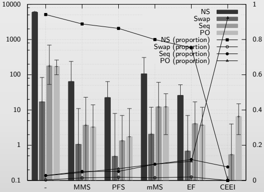

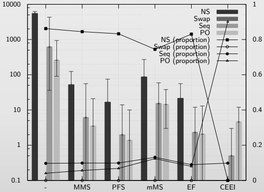

Take-away message

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Thank you

Want to see more?

http://recherche.noiraudes.net/en/cycle-deals.php

Pictures (shamefully) borrowed without permission from ADN (https://drawthesimpsons.tumblr.com/)

CEEI and efficiency

What we already know...

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Every CEEI allocation is Pareto-optimal if preferences are strict on shares.

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Price vector: $\langle 0.5, 1, 1, 0.5 \rangle$.

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