Fair Division under Ordinal Preferences: Computing Envy-Free Allocations of Indivisible Goods

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The fair division problem

Given

- a set of indivisible objects $O = \{o_1, \ldots, o_m\}$
- and a set of agents $A = \{1, \ldots, n\}$
- such that each agent has some preferences on the subsets of objects she may receive

Find

- an allocation $\pi : A \to 2^O$
- such that $\pi(i) \cap \pi(j)$ for every $i \neq j$
- satisfying some fairness and efficiency criteria

Preferences

- *cardinal*: agent i has a utility function $u_i : 2^O \to \mathbb{R}$
- *ordinal*: agent i has a preference relation $\succeq_i$ on 2^O

- ordinal preferences are easier to elicit
- ordinality does not require preferences to be interpersonally comparable
- several important criteria need only ordinal preferences

Ordinal preferences

- $\succeq$ weak order (transitive, reflexive and complete relation) on 2^{O}
- $A \succeq B$: the agent likes A at least as much as B
- strict preference: $A \succ B \Rightarrow A \succeq B$ and not $B \succeq A$
- indifference: $A \sim B \Rightarrow A \succeq B$ and $B \succeq A$

- Preferences over sets of goods are typically *monotonic*: $A \supseteq B \Rightarrow A \succeq B$.

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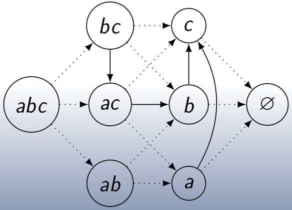

Compact preference representation

- $O = \{o_1, \ldots, o_m\}$
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  needs **exponential** space

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2. restriction on the set of preferences an agent can express (examples: separable preference relations, additive utility functions);
3. compact representation languages

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Separable ordinal preferences

Restriction: an agent specifies a linear order $\rhd$ on $\mathcal{O}$ (**single** objects)

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How to lift $\rhd$ to a (partial) strict order $\succ_{\mathcal{N}}$ on 2^O ?

Take the smallest strict order that

- extends $\rhd$
- is (strictly) monotonic: if $X \supset Y$ then $X \succ Y$.
- is separable: if $(X \cup Y) \cap Z = \varnothing$ then $X \succ Y$ iff $X \cup Z \succ Y \cup Z$

Brams, S. J., Edelman, P. H., and Fishburn, P. C. (2004).
Fair division of indivisible items.
Theory and Decision, 5(2):147–180.

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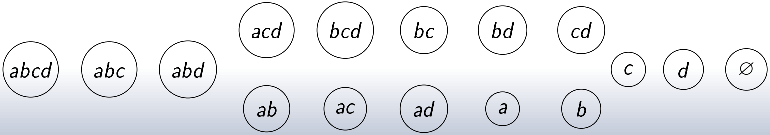
Example

- $\mathcal{N} : a \rhd b \rhd c \rhd d$
- Separability
- Monotonicity



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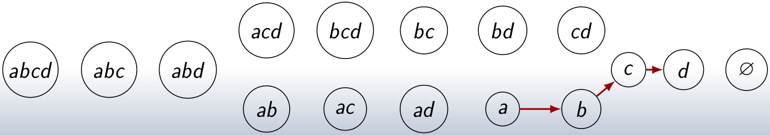

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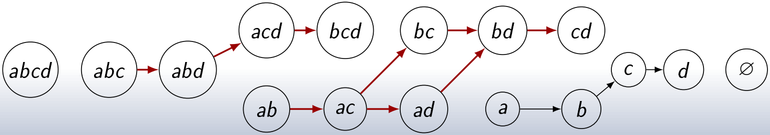

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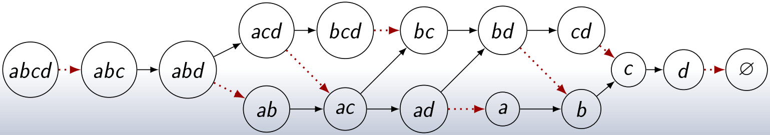

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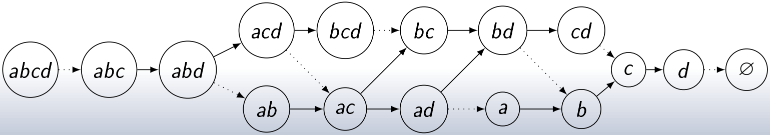

Dominance

An equivalent characterization

$A \succ_{\mathcal{N}} B$ *iff* there exists an injective mapping $g : B \to A$ such that $g(a) \trianglerighteq_{\mathcal{N}} a$ for all $a \in B$ and $g(a) \triangleright_{\mathcal{N}} a$ for some $a \in B$.

Example: $\mathcal{N} = a \triangleright b \triangleright c \triangleright d \triangleright e \triangleright f$

- $\{ a , c , d \} \succ_{\mathcal{N}} \{ b , c , e \}$
- $\{ a , d , e \}$ and $\{ b , c , f \}$ are incomparable.
- $\{a, c, d\}$ and $\{b, c, e, f\}$ are incomparable.

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Fair division

From individual preferences to collective resource allocation...

- Each agent has a linear order on O.
- We lift these orders to strict partial orders on 2^O.
- We look for a **fair** and **efficient** allocation.

Envy-freeness

Fairness...

Envy-freeness

Fairness…

Classical envy-freeness

Given a profile $\langle \succ_1, \ldots, \succ_n \rangle$ of **total** strict orders, an allocation π is *envy-free* if for all i, j, $\pi(i) \succ_i \pi(j)$.

When $\langle \succ_1, \ldots, \succ_n \rangle$ are partial orders ?

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$\leadsto$ Envy-freeness becomes a **modal** notion

Possible and necessary Envy-freeness

- π is Possibly Envy-Free *iff* for all i, j, we have $\pi(j) \not\succ_i \pi(i)$;
- π is Necessary Envy-Free *iff* for all i, j, we have $\pi(i) \succ_i \pi(j)$.

Envy-freeness

4 goods, 2 agents

- $\mathcal{N}_1 = a \triangleright b \triangleright c \triangleright d$
- $\mathcal{N}_2 = d \triangleright c \triangleright b \triangleright a.$

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- $\mathcal{N}_1 = a \rhd b \rhd c \rhd d$
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- $\pi : 1 \mapsto \{a, d\}; 2 \mapsto \{b, c\}$.

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  - π is not NEF

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  - π is not NEF
- $\pi' : 1 \mapsto \{a, b\}; 2 \mapsto \{c, d\}.$

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  - π is not NEF
- $\pi' : 1 \mapsto \{a, b\}; 2 \mapsto \{c, d\}$.
  - π' is NEF

Pareto-efficiency

Efficiency...

Pareto-efficiency

Efficiency…

- Complete allocation.
- Pareto-efficiency

Pareto-efficiency

Efficiency…

Classical Pareto dominance

π' **dominates** π if for all i, $\pi'(i) \succeq_i \pi(i)$ and for some j, $\pi'(j) \succ_j \pi(j)$

Possible and necessary Pareto dominance

- π' possibly dominates π *iff* for all i, $\pi(i) \not\succ_i \pi'(i)$ and for some j, we have $\pi(j) \not\succeq_j \pi'(j)$;
- π' necessarily dominates π *iff* for all i, $\pi'(i) \succeq_i \pi(i)$ and for some j, we have $\pi'(j) \succ_j \pi(j)$.

- π is *possibly Pareto-efficient* (PPE) if there exists no allocation π' such that π' necessarily dominates π.
- π' is *necessarily Pareto-efficient* (NPE) if there exists no allocation π' such that π' possibly dominates π.

Envy-freeness and efficiency

complete PPE NPE ← Efficiency

Envy-freeness and efficiency

complete PPE NPE ← Efficiency

PEF
NEF
↑
Fairness

Envy-freeness and efficiency

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

Efficiency (→ complete, PPE, NPE)

Fairness (→ PEF, NEF)

Envy-freeness and efficiency

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

Efficiency

Fairness

Envy-freeness and efficiency cannot always be satisfied simultaneously

Envy-freeness and efficiency

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

Efficiency (columns) — Fairness (rows)

Envy-freeness and efficiency cannot always be satisfied simultaneously

Questions:

- under which conditions is it guaranteed that there exists a allocation that satisfies Fairness and Efficiency ?
- how hard it is to determine whether such an allocation exists?

Complete possibly envy-free allocations

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

Complete possibly envy-free allocations

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

Result

If n agents express their preferences over m goods and k distinct goods are top-ranked by some agent, then there exists a complete PEF allocation if and only if $m \geq 2n - k$.

Constructive proof (algorithm/protocol)

Example

$\mathcal{N}_1$: $a \rhd b \rhd c \rhd d \rhd e \rhd f$ $\quad$ $\mathcal{N}_2$: $a \rhd d \rhd b \rhd c \rhd e \rhd f$

$\mathcal{N}_3$: $b \rhd a \rhd c \rhd d \rhd f \rhd e$ $\quad$ $\mathcal{N}_4$: $b \rhd a \rhd c \rhd e \rhd f \rhd d$

$$(k = 2;\ m = 6 \geq 2n - k)$$

Consider the agents in order $1 > 2 > 3 > 4$:

- *first step*: give a to 1; give b to 3;
- *second step*: give d to 2; give c to 4;
- *third step*: give e to 4; give f to 2.

PPE-PEF allocations

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

PPE-PEF allocations

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PEF	X	X	X
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Result

There exists a PPE-PEF allocation if and only if there exists a complete, PEF allocation.

PPE-PEF allocations

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

Result

There exists a PPE-PEF allocation if and only if there exists a complete, PEF allocation.

Key point of the proof: the previous protocol can be refined into a protocol returning an allocation that is the product of *sincere choices* by the agents in some sequence, and then use a result from [Brams and King, 2005].

Brams, S. J. and King, D. (2005).
Efficient fair division—help the worst off or avoid envy?
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NPE-PEF allocations

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PEF	X	X	X
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NPE-PEF allocations

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

Complexity of the existence of NPE-PEF allocations: *open*.

Complete NEF allocations

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

Complete NEF allocations

	complete	PPE	NPE
PEF	X	X	X
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Two necessary conditions:

- the number m of goods must be a multiple of the n number of agents.
- the top objects must be all distinct

Complete NEF allocations

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

Two necessary conditions:

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Complete allocation

- deciding whether there exists a complete NEF allocation is NP-complete (even if $m = 2n$).
- the problem falls down in P for two agents

(hardness by reduction from [X3C])

Pareto-efficient-NEF allocations

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

Pareto-efficient-NEF allocations

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

Possible and necessary Pareto-efficiency

- existence of a PPE-NEF allocation: NP-complete
- existence of a NPE-NEF allocation: NP-hard but probably not in NP (Σ_2^p-completeness conjectured).

Complexity results

	complete	**PPE**	**NPE**
PEF	P	P	?
NEF	NP-complete	NP-complete (P for 2 agents)	NP-hard (Σ_2^p-completeness conjectured)

Framework and results

Fair division with incomplete ordinal preferences:

- separable and monotone ordinal preferences;
- modal Pareto-efficiency and Envy-freeness.

Results: fair division (Efficient and Envy-Free allocation) not tractable (NP-hard) in general.

Future work

Beyond separable preferences ? CI-nets [Bouveret et al., 2009].
$\rightsquigarrow$ Even dominance is PSPACE-complete.

Solutions ?

- $\Rightarrow$ other fairness criteria (than envy-freeness);
- $\Rightarrow$ other tractable fragments (than SCI-nets);
- $\Rightarrow$ approximate dominance relation.

Bouveret, S., Endriss, U., and Lang, J. (2009).
Conditional importance networks: A graphical language for representing ordinal, monotonic preferences over sets of goods.
In *Proceedings of the 21st International Joint Conference on Artificial Intelligence (IJCAI'09)*, pages 67–72, Pasadena, California.