Algorithmes de programmation par contraintes pour la recherche d'allocations leximin-optimales

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Fairness in combinatorial problems. . .

Many real-world combinatorial problems. . .

- Nurse rostering problem.
- Balanced timetables.
- Fair allocation of airport and airspace resources (to several airlines).
- Fair share of Earth Observation Satellites.

. . . imply human **agents** and thus involve directly or indirectly the concept of **fairness**.

All of these problems are combinatorial collective decision making problems, under admissibility constraints, and fairness requirements.

How can we model and solve this kind of problems in a Constraint Programming framework ?

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Outline

1. Formalizing and modeling the problem
   - CSP and constraint programming
   - Fairness in collective decision making
   - Leximin and Constraint Programming
2. Two constraint programming algorithms
   - Using a set of cardinality constraints
   - Using a multiset ordering constraint
   - Sort and conquer
   - Using the saturated subsets
3. Application, tests and results
   - Benchmark and results

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Constraint networks

Constraint network [Montanari, 1974]

A constraint network is based on :

- a set of variables $\mathcal{X} = \{x_1, \ldots, x_p\}$;
- a set of domains $\mathcal{D} = \{d_{x_1}, \ldots, d_{x_p}\}$;
- a set of constraints $\mathcal{C}$, with, for all $C \in \mathcal{C}$:
  - $X(C)$ the scope of the constraint,
  - $R(C)$ the set of allowed tuples of the constraint.

Montanari, U. (1974).
Networks of constraints: Fundamental properties and applications to picture processing.
Information Sciences, 7:95–132.

The Constraint Satisfaction Problem

Classical CSP

Given : A constraint network $(\mathcal{X}, \mathcal{D}, \mathcal{C})$.
Is there a complete consistent instantiation v *of* $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ *?*

$\rightsquigarrow$ **NP**-complete.

CSP with objective variable

Given : A constraint network $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ and an objective variable $o \in \mathcal{X}$, such that $d_o \subset \mathbb{N}$.
What is the maximal value α *of* d_o *such that there is a complete consistent instantiation* $\widehat{v}$ *with* $\widehat{v}(o) = \alpha$ *?*

$\rightsquigarrow$ **NP**-complete (for the associated decision problem).

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Is there a complete consistent instantiation v of $(\mathcal{X}, \mathcal{D}, \mathcal{C})$?

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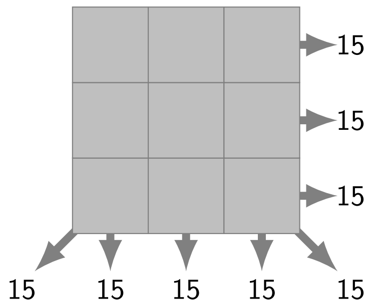
Example of a CSP

The magic square, aka. *Lo Shu* square



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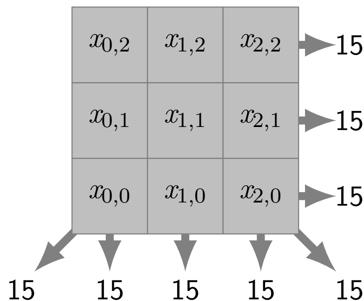

Variables:

x_i, j, with $(i, j) \in \{0, 1, 2\}^2$.

Domains:

$\forall i, j, \mathcal{D}_{x_{i,j}} = \{1, \dots, 9\}$.

Constraints:

$\textbf{alldiff}(x_{i,j} | (i, j) \in \{0, 1, 2\}^2)$,
$\forall i, \sum_{j=0}^{2} x_{i,j} = 15$,
$\forall j, \sum_{i=0}^{2} x_{i,j} = 15$,
$x_{0,0} + x_{1,1} + x_{2,2} = 15$,
$x_{2,0} + x_{1,1} + x_{0,2} = 15$.

Solving CSP with Constraint Programming

Constraint Programming is a solving framework for dealing with Constraint Satisfaction Problems. It is based on:

- a search algorithm,
- some contraint propagation mechanisms reacting on some events on the variables (bounds updated, instantiation, . . .).

Coming back on the example

$x_{0,2}$	$x_{1,2}$	$x_{2,2}$
$x_{0,1}$	$x_{1,1}$	$x_{2,1}$
$x_{0,0}$	$x_{1,0}$	$x_{2,0}$

$x_{0,0}$: $\{1,2,3,4,5,6,7,8,9\}$
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Events queue:

Coming back on the example

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awakeOnInf($\mathbf{alldiff}(x_{i,j} | (i,j) \in \{0,1,2\}^2)$, $x_{1,0}$) $\rightsquigarrow$ **Inconsistent!**,

...

Individual utility and collective decision

Given a collective decision making problem, the preferences of an agent regarding the set of admissible decisions $\mathcal{S}$ is given by a **utility function**:

Individual utility function

Given an agent a_i and the set of admissible decisions $\mathcal{S}$, the individual utility function of a_i is a function $u_i : \mathcal{S} \rightarrow \mathbb{N}$.

The quality of a collective decision is measured using its associated **utility profile**, that is, the vector of its associated utilities.

"[Welfarism] judges a collective action on the basis of the utility levels enjoyed by the individual agents, and on those levels only" [Moulin, 1988]

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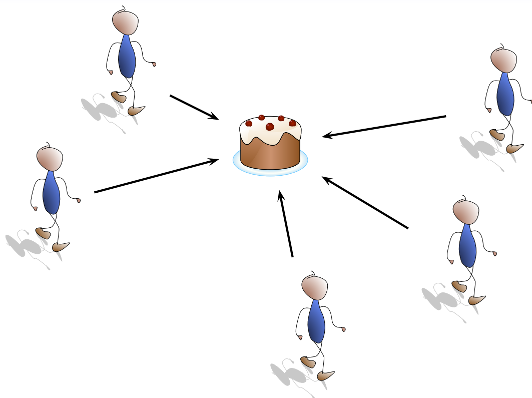
Decision making and *welfarism*



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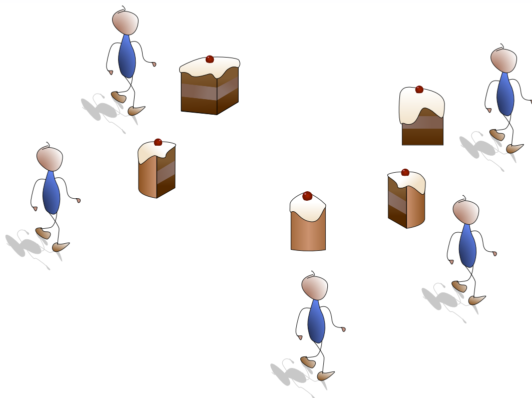

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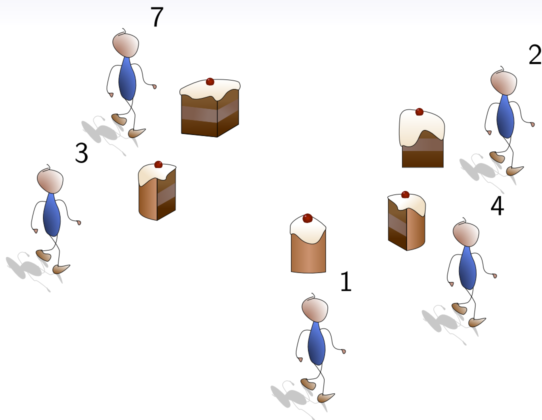

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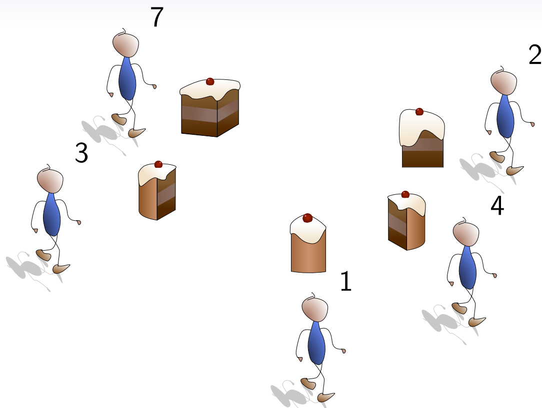

Utility profile: $\overrightarrow{u} = \langle 3, 4, 2, 1, 7 \rangle$

Social Welfare Ordering

We make use of a **Social Welfare Ordering** to compare the utility profiles.

Social Welfare Ordering

A **Social Welfare Ordering** is a preorder $\preceq$ on $\mathbb{N}^n$.

Examples

- Classical utilitarian Social Welfare Ordering.
- Egalitarian Social Welfare Ordering.
- Leximin Social Welfare Ordering.

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- Leximin SWO.

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- Egalitarian SWO.
- Leximin SWO.

Classical utilitarian SWO [Harsanyi]

$\overrightarrow{u} \preceq \overrightarrow{v} \Leftrightarrow \sum_{i=1}^{n} u_i \leq \sum_{i=1}^{n} v_i$.

Features

The agents are utility "producers".
It is indifferent to inequalities between agents $\leadsto$ it can lead to very unfair decisions.

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Fairness and utilitarian SWO

$\langle 10, 10, 10, 10 \rangle \preceq \langle 41, 0, 0, 0 \rangle$, whereas $\langle 10, 10, 10, 10 \rangle$ is more equitable than $\langle 41, 0, 0, 0 \rangle$.

Classical Social Welfare Orderings

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- Egalitarian SWO.
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Egalitarian SWO [Rawls]

$\overrightarrow{u} \preceq \overrightarrow{v} \Leftrightarrow \min_{i=1}^{n} u_i \leq \min_{i=1}^{n} v_i$.

Features

It only takes the least satisfied agent into account $\leadsto$ natural inclination to fairness.

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Egalitarian SWO [Rawls]

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Features

It only takes the least satisfied agent into account $\leadsto$ natural inclination to fairness.
On the other hand, it can lead to non Pareto-optimal decisions (drowning effect).

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Egalitarian SWO [Rawls]

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Egalitarian SWO and Pareto-efficiency

$\langle 1, 1, 1, 1 \rangle \sim \langle 1000, 1, 1000, 1000 \rangle$, whereas $\langle 1, 1, 1, 1 \rangle$ and $\langle 1000, 1, 1000, 1000 \rangle$ are very different !

Classical Social Welfare Orderings

- Classical utilitarian SWO.
- Egalitarian SWO.
- Leximin SWO.

Leximin SWO

Let $\overrightarrow{x}$ be a vector. We write $\overrightarrow{x}^{\uparrow}$ the sorted version of $\overrightarrow{x}$.
$\overrightarrow{u} \succ_{leximin} \overrightarrow{v} \Leftrightarrow \exists k$ such that $\forall i \leq k,\ u_i^{\uparrow} = v_i^{\uparrow}$ and $u_{k+1}^{\uparrow} > v_{k+1}^{\uparrow}$.
In other words, a lexicographic comparison on the sorted vectors.

Performing a leximin comparison...

Two vectors to compare: $\overrightarrow{u} = \langle 4, 10, 3, 5 \rangle$ and $\overrightarrow{v} = \langle 4, 3, 6, 6 \rangle$.

- We sort the vectors: $\begin{cases} \overrightarrow{u}^{\uparrow} = \langle 3, 4, 5, 10 \rangle \\ \overrightarrow{v}^{\uparrow} = \langle 3, 4, 6, 6 \rangle \end{cases}$
- We compare the sorted vectors lexicographically: $\overrightarrow{u}^{\uparrow} \prec_{lexico} \overrightarrow{v}^{\uparrow}$

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In other words, a lexicographic comparison on the sorted vectors.

Features

It takes all agents into account, in the order of their satisfaction level $\leadsto$ natural inclination to fairness.
It both refines the order induced by the egalitarian SWO and the Pareto order.

Classical Social Welfare Orderings

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Leximin SWO

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In other words, a lexicographic comparison on the sorted vectors.

Leximin SWO and Pareto-efficiency

$\langle 1, 1, 1, 1 \rangle \prec \langle 1000, 1, 1000, 1000 \rangle$ (the second value of the ordered vectors is discriminatory).

The Constraint Satisfaction Problem

Classical CSP

Given : A constraint network $(\mathcal{X}, \mathcal{D}, \mathcal{C})$.
Is there a complete consistent instantiation v of $(\mathcal{X}, \mathcal{D}, \mathcal{C})$?

CSP with objective variable

Given : A constraint network $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ and an objective variable $o \in \mathcal{X}$, such that $d_o \subset \mathbb{N}$.
What is the maximal value α of d_o such that there is a complete consistent instantiation $\widehat{v}$ with $\widehat{v}(o) = \alpha$?

Leximin-CSP (as a multi-objective CSP)

Given : A constraint network $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ and a vector of variables $\overrightarrow{u} = \langle u_1, \ldots, u_n \rangle$ ($\forall i,\ u_i \in \mathcal{X}$ and $d_{u_i} \in \mathbb{N}$) called **objective vector**.
What is the leximin-optimal vector $\langle \alpha_1, \ldots, \alpha_n \rangle$ of $\langle d_{u_1}, \ldots, d_{u_n} \rangle$ such that there is a complete consistent instantiation $\widehat{v}$ with $\widehat{v}(u_i) = \alpha_i$ forall i ?

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Algorithm 1

Using some cardinality constraints

An alternative definition of sorting

Proposition

$\langle y_1, \ldots, y_n \rangle$ is the permutation of $\langle u_1, \ldots, u_n \rangle$ sorted in non-decreasing order if and only if:

- y_1 is the maximal value such that all the u_i are g.e.q than y_1;
- y_2 is the maximal value such that at least $n-1$ values among the u_i are g.e.q than y_2 and all the u_i are g.e.q than y_1;

$$\vdots$$

- y_n is the maximal value such that at least 1 value among the u_i is g.e.q than y_n, at least 2 values among the u_i are g.e.q than y_{n-1}, $\ldots$, and all the u_i are g.e.q than y_1.

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Proposition

$\langle y_1, \ldots, y_n \rangle$ is the permutation of $\langle u_1, \ldots, u_n \rangle$ sorted in non-decreasing order if and only if:

- y_1 is the maximal value such that all the u_i are g.e.q than y_1;
- y_2 is the maximal value such that at least $n-1$ values among the u_i are g.e.q than y_2 and all the u_i are g.e.q than y_1;

$$\vdots$$

- y_n is the maximal value such that at least 1 value among the u_i is g.e.q than y_n, at least 2 values among the u_i are g.e.q than y_{n-1}, $\ldots$, and all the u_i are g.e.q than y_1.

An alternative definition of sorting

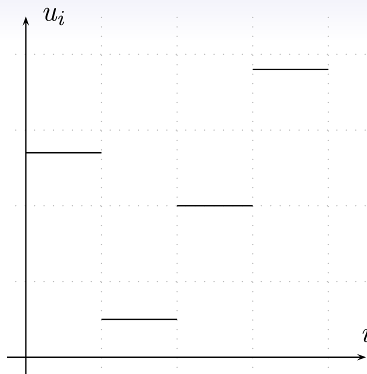
Proposition

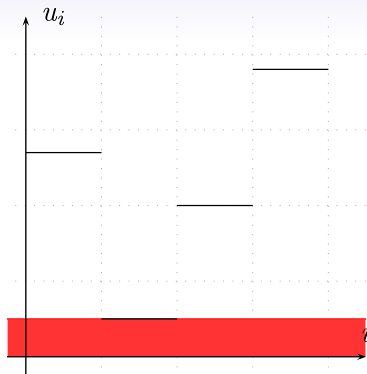
$\langle y_1, \ldots, y_n \rangle$ is the permutation of $\langle u_1, \ldots, u_n \rangle$ sorted in non-decreasing order if and only if:

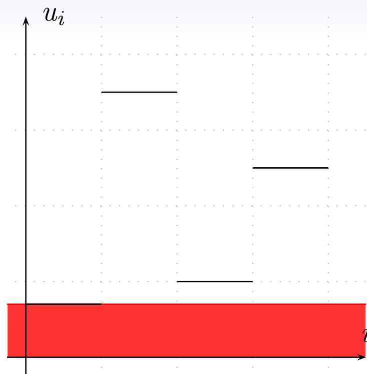
- y_1 is the maximal value such that all the u_i are g.e.q than y_1;
- y_2 is the maximal value such that at least $n-1$ values among the u_i are g.e.q than y_2 and all the u_i are g.e.q than y_1;

$$\vdots$$

- y_n is the maximal value such that at least 1 value among the u_i is g.e.q than y_n, at least 2 values among the u_i are g.e.q than y_{n-1}, $\ldots$, and all the u_i are g.e.q than y_1.


u_i
i


u_i
i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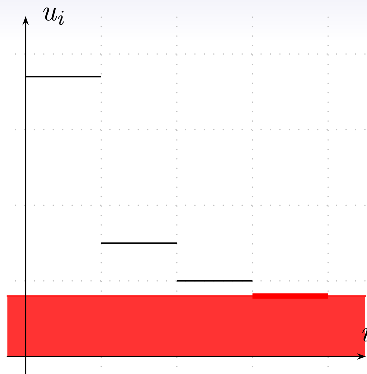
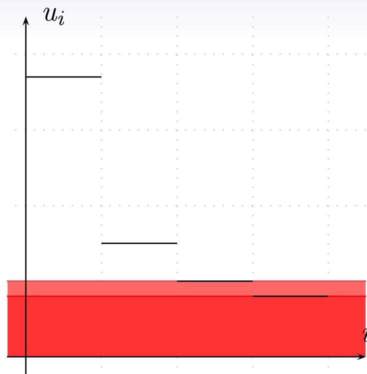
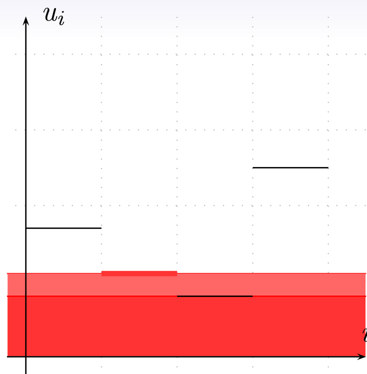
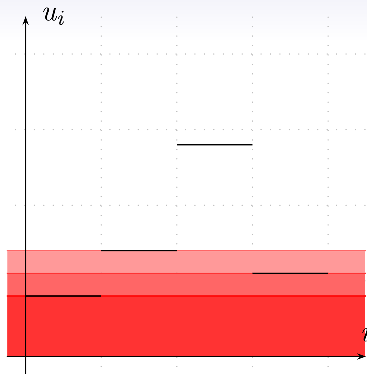


u_i
i


u_i
i


u_i
i


u_i
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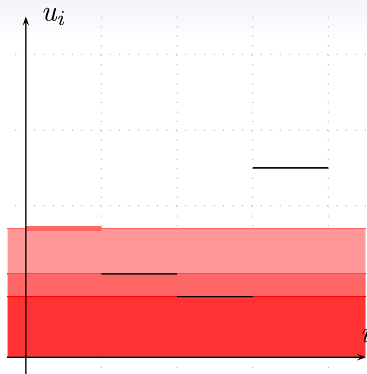
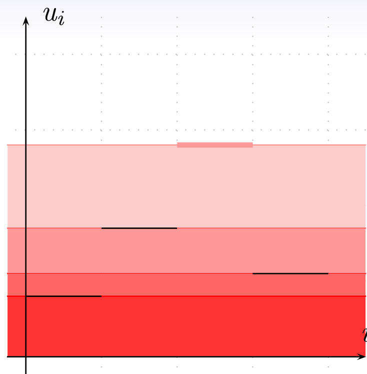

u_i
i


u_i
i

The meta-constraint AtLeast

The algorithm is based on the meta-constraint **AtLeast**

Constraint **AtLeast** [Van Hentenryck et al., 1992]

Let Γ be a set of p constraints, and $k \in [\![1, p]\!]$ be an integer. The meta-constraint $\mathbf{AtLeast}(\Gamma, k)$ is the constraint that holds on the union of the scopes of the constraints in Γ, and that allows a tuple if and only if at least k constraints from Γ are satisfied.

Van Hentenryck, P., Simonis, H., and Dincbas, M. (1992).
Constraint satisfaction using constraint logic programming.
A.I., 58(1-3):113–159.

Algorithm description

Input: $(\mathcal{X}, \mathcal{D}, \mathcal{C})$; $\langle u_1, \ldots, u_n \rangle \in \mathcal{X}^n$ objective variables (utilities of agents)
Output: The leximin-optimal instantiation or "Inconsistent"

Algorithm

if $\text{solve}(\mathcal{X}, \mathcal{D}, \mathcal{C}) =$ **"Inconsistent" then return "Inconsistent"**

$\mathcal{X}' \leftarrow \mathcal{X} \cup \{y_1, \ldots, y_n\}$;
$\mathcal{D}' \leftarrow \mathcal{D} \cup \{[\![m, M]\!], \ldots, [\![m, M]\!]\}$;
$\mathcal{C}' \leftarrow \mathcal{C}$;

for $i \leftarrow 1$ **to** n **do**
  $\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n - i + 1)\}$;
  $\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$;
  $d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$;
end
return $\widehat{v}_{\downarrow \mathcal{X}}$

Algorithm description

Input: $(\mathcal{X}, \mathcal{D}, \mathcal{C})$; $\langle u_1, \ldots, u_n \rangle \in \mathcal{X}^n$ objective variables (utilities of agents)
Output: The leximin-optimal instantiation or "Inconsistent"

Algorithm

if $\text{solve}(\mathcal{X}, \mathcal{D}, \mathcal{C}) =$ **"Inconsistent" then return "Inconsistent"**

Introduction of the variables $\mathrm{y_i}$

$\mathcal{X}' \leftarrow \mathcal{X} \cup \{y_1, \ldots, y_n\}$;
$\mathcal{D}' \leftarrow \mathcal{D} \cup \{[\![m, M]\!], \ldots, [\![m, M]\!]\}$;
$\mathcal{C}' \leftarrow \mathcal{C}$;

for $i \leftarrow 1$ **to** n **do**
  $\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n - i + 1)\}$;
  $\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$;
  $d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$;
end
return $\widehat{v}_{\downarrow \mathcal{X}}$

Algorithm description

Input: $(\mathcal{X}, \mathcal{D}, \mathcal{C})$; $\langle u_1, \ldots, u_n \rangle \in \mathcal{X}^n$ objective variables (utilities of agents)
Output: The leximin-optimal instantiation or "Inconsistent"

Algorithm

if solve$(\mathcal{X}, \mathcal{D}, \mathcal{C})$ = **"Inconsistent" then return "Inconsistent"**

Introduction of the variables y_i

$\mathcal{X}' \leftarrow \mathcal{X} \cup \{y_1, \ldots, y_n\}$;
$\mathcal{D}' \leftarrow \mathcal{D} \cup \{[\![m, M]\!], \ldots, [\![m, M]\!]\}$;
$\mathcal{C}' \leftarrow \mathcal{C}$;

Introduction of a constraint on y_i

for $i \leftarrow 1$ **to** n **do**
  $\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n - i + 1)\}$;
  $\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$;
  $d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$;
end
return $\widehat{v}_{\downarrow \mathcal{X}}$

Algorithm description

Input: $(\mathcal{X}, \mathcal{D}, \mathcal{C})$; $\langle u_1, \ldots, u_n \rangle \in \mathcal{X}^n$ objective variables (utilities of agents)
Output: The leximin-optimal instantiation or "Inconsistent"

Algorithm

if $\text{solve}(\mathcal{X}, \mathcal{D}, \mathcal{C}) =$ **"Inconsistent" then return "Inconsistent"**

Introduction of the variables $\mathrm{y_i}$

$\mathcal{X}' \leftarrow \mathcal{X} \cup \{y_1, \ldots, y_n\}$;
$\mathcal{D}' \leftarrow \mathcal{D} \cup \{[\![m, M]\!], \ldots, [\![m, M]\!]\}$;
$\mathcal{C}' \leftarrow \mathcal{C}$;

Introduction of a constraint on $\mathrm{y_i}$

for $i \leftarrow 1$ **to** n **do**

$\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n - i + 1)\}$;
$\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$; Computation of $\mathrm{y_i}$
$d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$;

end

return $\widehat{v}_{\downarrow \mathcal{X}}$

Algorithm description

Input: $(\mathcal{X}, \mathcal{D}, \mathcal{C})$; $\langle u_1, \ldots, u_n \rangle \in \mathcal{X}^n$ objective variables (utilities of agents)
Output: The leximin-optimal instantiation or "Inconsistent"

Algorithm

if $\text{solve}(\mathcal{X}, \mathcal{D}, \mathcal{C})$ = **"Inconsistent" then return "Inconsistent"**

Introduction of the variables y_i

$\mathcal{X}' \leftarrow \mathcal{X} \cup \{y_1, \ldots, y_n\}$;
$\mathcal{D}' \leftarrow \mathcal{D} \cup \{[\![m, M]\!], \ldots, [\![m, M]\!]\}$;
$\mathcal{C}' \leftarrow \mathcal{C}$;

Introduction of a constraint on y_i

for $i \leftarrow 1$ **to** n **do**

$\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n - i + 1)\}$;

$\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$; — Computation of y_i

$d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$; — Updating of the domains

end

return $\widehat{v}_{\downarrow \mathcal{X}}$

Running of the algorithm on an example

A sharing problem

- An allocation problem with 3 agents and 3 objects.
- Constraint: the allocation has to be a matching (*i.e.* one different object per agent).
- The utility functions of the agents are defined by a set of weights $w(a_i, o_j)$, the utility function of the agent a_i being $u_i = \sum_{o_j | a_i \leftarrow o_j} w(a_i, o_j)$.
- The weights are the following:

objects \ agents	a_1	a_2	a_3
o_1	3	3	3
o_2	5	9	7
o_3	7	8	1

Running of the algorithm on an example

Algorithm :

$\mathcal{X}' \leftarrow \mathcal{X} \cup \{y_1, \ldots, y_n\}$;

$\mathcal{D}' \leftarrow \mathcal{D} \cup \{[\![m, M]\!], \ldots, [\![m, M]\!]\}$;

$\mathcal{C}' \leftarrow \mathcal{C}$;

Initial constraint network:

$\mathcal{X}' = \{a_1, a_2, a_3, u_1, u_2, u_3\} \cup \{\mathbf{y_1}, \mathbf{y_2}, \mathbf{y_3}\}$

$\mathcal{D}' = \{\{o_1, o_2, o_3\}, \{o_1, o_2, o_3\}, \{o_1, o_2, o_3\}, [\![3, 7]\!], [\![3, 9]\!], [\![1, 7]\!]\}$
$\cup \{[\![\mathbf{1}, \mathbf{9}]\!], [\![\mathbf{1}, \mathbf{9}]\!], [\![\mathbf{1}, \mathbf{9}]\!]\}$

$\mathcal{C}' = \{\textbf{alldifferent}(\{a_1, a_2, a_3\}), u_i = \sum_{o_j} (a_i = o_j)\ \forall i\}$

Running of the algorithm on an example

Algorithm :

for $i \leftarrow 1$ **to** n **do**

$\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n - i + 1)\}$;

$\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$;

$d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$;

end

a_1	a_2	a_3	u_1	u_2	u_3	y_1	y_2	y_3
o_1	o_2	o_3	3	9	1	$\{1,\ldots,9\}$	$\{1,\ldots,9\}$	$\{1,\ldots,9\}$
o_1	o_3	o_2	3	8	7	$\{1,\ldots,9\}$	$\{1,\ldots,9\}$	$\{1,\ldots,9\}$
o_2	o_1	o_3	5	3	1	$\{1,\ldots,9\}$	$\{1,\ldots,9\}$	$\{1,\ldots,9\}$
o_2	o_3	o_1	5	8	3	$\{1,\ldots,9\}$	$\{1,\ldots,9\}$	$\{1,\ldots,9\}$
o_3	o_1	o_2	7	3	7	$\{1,\ldots,9\}$	$\{1,\ldots,9\}$	$\{1,\ldots,9\}$
o_3	o_2	o_1	7	9	3	$\{1,\ldots,9\}$	$\{1,\ldots,9\}$	$\{1,\ldots,9\}$

Running of the algorithm on an example

Algorithm :

for $i \leftarrow 1$ **to** n **do**

$\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n - i + 1)\}$;

$\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$;

$d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$;

end

a_1	a_2	a_3	u_1	u_2	u_3	y_1	y_2	y_3
o_1	o_2	o_3	3	9	1	**1**	{1,...,9}	{1,...,9}
o_1	o_3	o_2	3	8	7	**{1,2,3}**	{1,...,9}	{1,...,9}
o_2	o_1	o_3	5	3	1	**1**	{1,...,9}	{1,...,9}
o_2	o_3	o_1	5	8	3	**{1,2,3}**	{1,...,9}	{1,...,9}
o_3	o_1	o_2	7	3	7	**{1,2,3}**	{1,...,9}	{1,...,9}
o_3	o_2	o_1	7	9	3	**{1,2,3}**	{1,...,9}	{1,...,9}

Running of the algorithm on an example

Algorithm :

for $i \leftarrow 1$ **to** n **do**

$\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n - i + 1)\}$;

$\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$;

$d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$;

end

a_1	a_2	a_3	u_1	u_2	u_3	y_1	y_2	y_3
o_1	o_2	o_3	3	9	1	1	{1,. . . ,9}	{1,. . . ,9}
o_1	**o_3**	**o_2**	**3**	**8**	**7**	{1,2,**3**}	**{1,. . . ,9}**	**{1,. . . ,9}**
o_2	o_1	o_3	5	3	1	1	{1,. . . ,9}	{1,. . . ,9}
o_2	**o_3**	**o_1**	**5**	**8**	**3**	{1,2,**3**}	**{1,. . . ,9}**	**{1,. . . ,9}**
o_3	**o_1**	**o_2**	**7**	**3**	**7**	{1,2,**3**}	**{1,. . . ,9}**	**{1,. . . ,9}**
o_3	**o_2**	**o_1**	**7**	**9**	**3**	{1,2,**3**}	**{1,. . . ,9}**	**{1,. . . ,9}**

Running of the algorithm on an example

Algorithm :

for $i \leftarrow 1$ **to** n **do**

$\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n - i + 1)\}$;

$\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$;

$d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$;

end

a_1	a_2	a_3	u_1	u_2	u_3	y_1	y_2	y_3
o_1	o_2	o_3	3	9	1	1	{1,...,9}	{1,...,9}
o_1	o_3	o_2	3	8	7	3	{1,...,9}	{1,...,9}
o_2	o_1	o_3	5	3	1	1	{1,...,9}	{1,...,9}
o_2	o_3	o_1	5	8	3	3	{1,...,9}	{1,...,9}
o_3	o_1	o_2	7	3	7	3	{1,...,9}	{1,...,9}
o_3	o_2	o_1	7	9	3	3	{1,...,9}	{1,...,9}

Running of the algorithm on an example

Algorithm :

for $i \leftarrow 1$ **to** n **do**

$\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n-i+1)\}$;

$\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$;

$d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$;

end

a_1	a_2	a_3	u_1	u_2	u_3	y_1	y_2	y_3
o_1	o_2	o_3	3	9	1	1	{1,...,9}	{1,...,9}
o_1	o_3	o_2	3	8	7	3	**{3,...,7}**	{1,...,9}
o_2	o_1	o_3	5	3	1	1	{1,...,9}	{1,...,9}
o_2	o_3	o_1	5	8	3	3	**{3,...,5}**	{1,...,9}
o_3	o_1	o_2	7	3	7	3	**{3,...,7}**	{1,...,9}
o_3	o_2	o_1	7	9	3	3	**{3,...,7}**	{1,...,9}

Running of the algorithm on an example

Algorithm :

for $i \leftarrow 1$ **to** n **do**

$\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n - i + 1)\}$;

$\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$;

$d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$;

end

a_1	a_2	a_3	u_1	u_2	u_3	y_1	y_2	y_3
o_1	o_2	o_3	3	9	1	1	{1,...,9}	{1,...,9}
$\mathbf{o_1}$	$\mathbf{o_3}$	$\mathbf{o_2}$	**3**	**8**	**7**	**3**	{3,...,**7**}	**{1,...,9}**
o_2	o_1	o_3	5	3	1	1	{1,...,9}	{1,...,9}
o_2	o_3	o_1	5	8	3	3	{3,...,5}	{1,...,9}
$\mathbf{o_3}$	$\mathbf{o_1}$	$\mathbf{o_2}$	**7**	**3**	**7**	**3**	{3,...,**7**}	**{1,...,9}**
$\mathbf{o_3}$	$\mathbf{o_2}$	$\mathbf{o_1}$	**7**	**9**	**3**	**3**	{3,...,**7**}	**{1,...,9}**

Running of the algorithm on an example

Algorithm :

for $i \leftarrow 1$ **to** n **do**

$\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n - i + 1)\}$;

$\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$;

$d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$;

end

a_1	a_2	a_3	u_1	u_2	u_3	y_1	y_2	y_3
o_1	o_2	o_3	3	9	1	1	{1,. . . ,9}	{1,. . . ,9}
o_1	o_3	o_2	3	8	7	3	**7**	{1,. . . ,9}
o_2	o_1	o_3	5	3	1	1	{1,. . . ,9}	{1,. . . ,9}
o_2	o_3	o_1	5	8	3	3	{3,. . . ,5}	{1,. . . ,9}
o_3	o_1	o_2	7	3	7	3	**7**	{1,. . . ,9}
o_3	o_2	o_1	7	9	3	3	**7**	{1,. . . ,9}

Running of the algorithm on an example

Algorithm :

for $i \leftarrow 1$ **to** n **do**

$\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n-i+1)\}$;

$\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$;

$d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$;

end

a_1	a_2	a_3	u_1	u_2	u_3	y_1	y_2	y_3
o_1	o_2	o_3	3	9	1	1	{1,...,9}	{1,...,9}
o_1	o_3	o_2	3	8	7	3	7	**{7,8}**
o_2	o_1	o_3	5	3	1	1	{1,...,9}	{1,...,9}
o_2	o_3	o_1	5	8	3	3	{3,...,5}	{1,...,9}
o_3	o_1	o_2	7	3	7	3	7	**7**
o_3	o_2	o_1	7	9	3	3	7	**{7,8,9}**

Running of the algorithm on an example

Algorithm :

for $i \leftarrow 1$ **to** n **do**

$\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n - i + 1)\}$;

$\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$;

$d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$;

end

a_1	a_2	a_3	u_1	u_2	u_3	y_1	y_2	y_3
o_1	o_2	o_3	3	9	1	1	{1,...,9}	{1,...,9}
o_1	o_3	o_2	3	8	7	3	7	{7,8}
o_2	o_1	o_3	5	3	1	1	{1,...,9}	{1,...,9}
o_2	o_3	o_1	5	8	3	3	{3,...,5}	{1,...,9}
o_3	o_1	o_2	7	3	7	3	7	7
$\mathbf{o_3}$	$\mathbf{o_2}$	$\mathbf{o_1}$	**7**	**9**	**3**	**3**	**7**	{7,8,**9**}

Running of the algorithm on an example

Algorithm :

for $i \leftarrow 1$ **to** n **do**

$\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{AtLeast}(\{u_1 \geq y_i, \ldots, u_n \geq y_i\}, n - i + 1)\}$;

$\widehat{v} \leftarrow \mathbf{maximize}(y_i, (\mathcal{X}, \mathcal{D}, \mathcal{C}))$;

$d_{y_i} \leftarrow \{\widehat{v}(y_i)\}$;

end

a_1	a_2	a_3	u_1	u_2	u_3	y_1	y_2	y_3
o_1	o_2	o_3	3	9	1	1	{1,...,9}	{1,...,9}
o_1	o_3	o_2	3	8	7	3	7	{7,8}
o_2	o_1	o_3	5	3	1	1	{1,...,9}	{1,...,9}
o_2	o_3	o_1	5	8	3	3	{3,...,5}	{1,...,9}
o_3	o_1	o_2	7	3	7	3	7	7
o_3	o_2	o_1	7	9	3	3	7	**9**

Algorithm 2

An algorithm inspired by a multiset ordering constraint

A second algorithm

We introduce a new constraint:

Constraint Leximin

Let $\overrightarrow{x}$ be a vector of variables and $\overrightarrow{\lambda}$ be a vector of integers. The constraint $\mathbf{Leximin}(\overrightarrow{\lambda}, \overrightarrow{x})$ concerns every variables belonging to $\overrightarrow{x}$, and allows a tuple $\overrightarrow{v(x)}$ if and only if $\overrightarrow{\lambda} \prec_{leximin} \overrightarrow{v(x)}$.

This constraint is based on the multiset ordering constraint, introduced in [Frisch et al., 2003] (algorithm to enforce GAC in $O(n \log(n))$).

Frisch, A., Hnich, B., Kiziltan, Z., Miguel, I., and Walsh, T. (2003).
Multiset ordering constraints.
In *Proc. of IJCAI'03*, Acapulco, Mexico.

Description of the algorithm

Algorithm

$v \leftarrow \mathbf{solve}(\mathcal{X}, \mathcal{D}, \mathcal{C})$

while $v \neq$ **"Inconsistent"** **do**

$\quad \widehat{v} \leftarrow v$

$\quad \mathcal{C} \leftarrow \mathcal{C} \cup \{\mathbf{Leximin}(\widehat{v}(\overrightarrow{u}), \overrightarrow{u})\}$

$\quad v \leftarrow \mathbf{solve}(\mathcal{X}, \mathcal{D}, \mathcal{C})$

end

if $\widehat{v} \neq \mathbf{null}$ **then** **return** $\widehat{v}$

else **return** **"Inconsistent"**

Algorithm 3

Sort and conquer

We introduce the sorted version of the objective vector

Idea : Introduce additional variables $\langle y_1, \ldots, y_n \rangle$ representing the sorted version of the objective vector, and perform successive maximizations.

1. Maximize y_1 : $\widehat{y_1}$.
2. Maximize y_2 under the constraint $y_1 = \widehat{y_1}$: $\widehat{y_2}$.

n. Maximize y_n under the constraints $y_1 = \widehat{y_1}, \ldots, y_{n-1} = \widehat{y_{n-1}}$.

We introduce the sorted version of the objective vector

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$$\vdots$$

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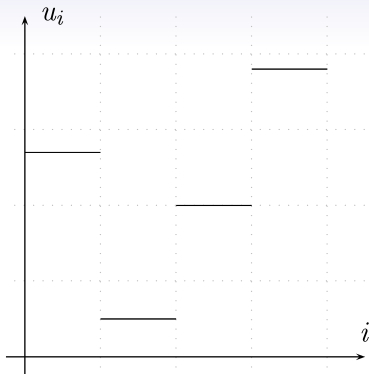
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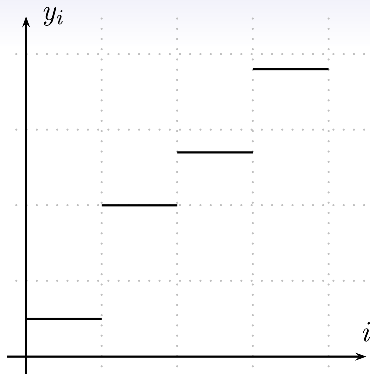
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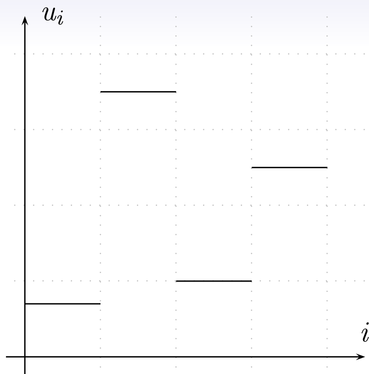
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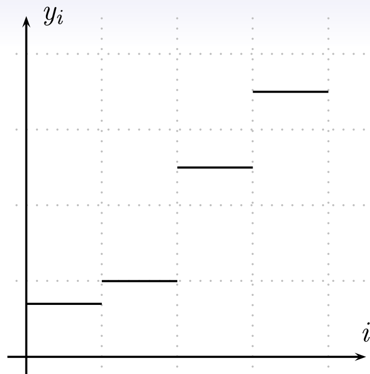
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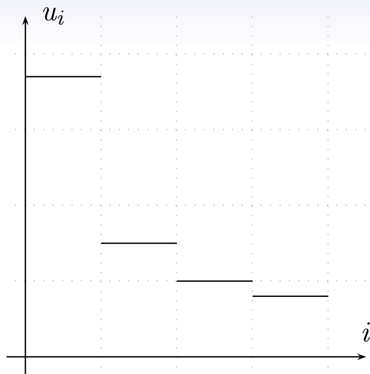
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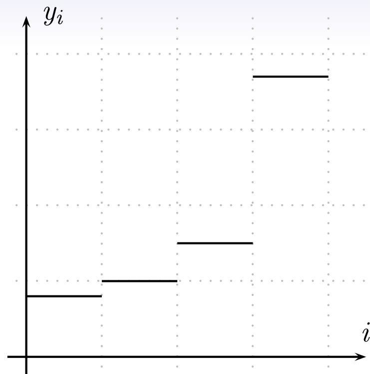
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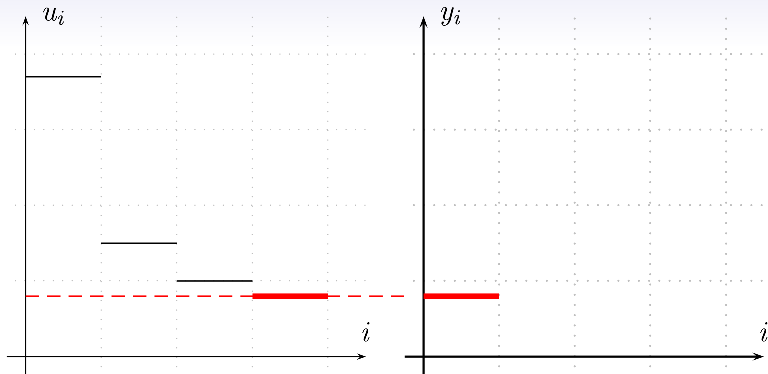

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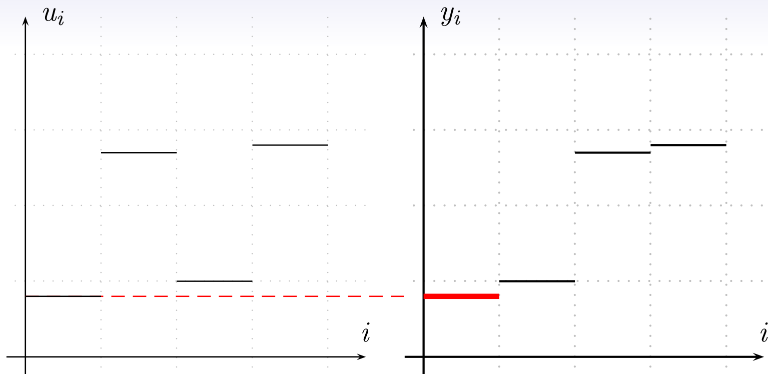
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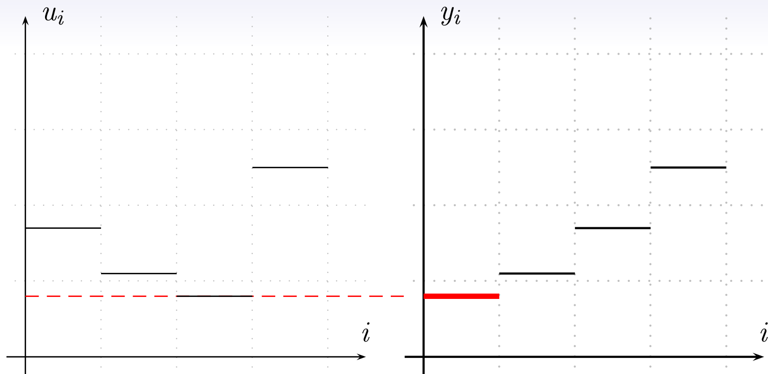
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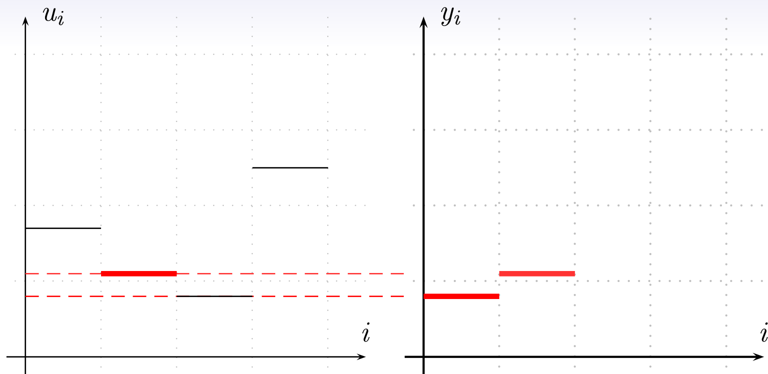
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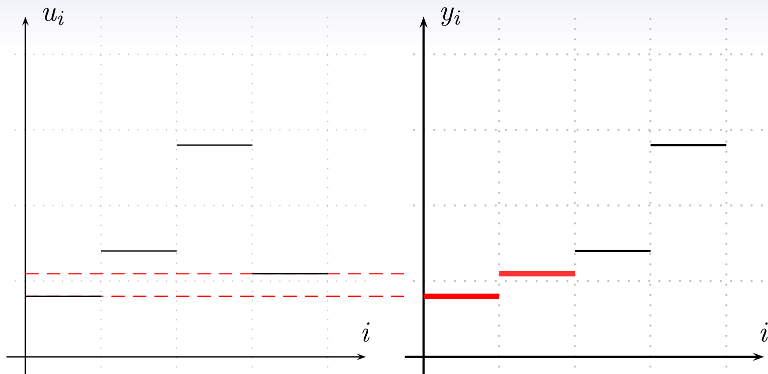
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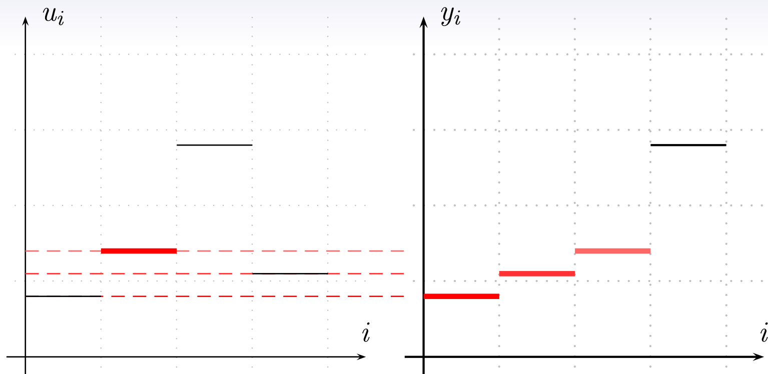
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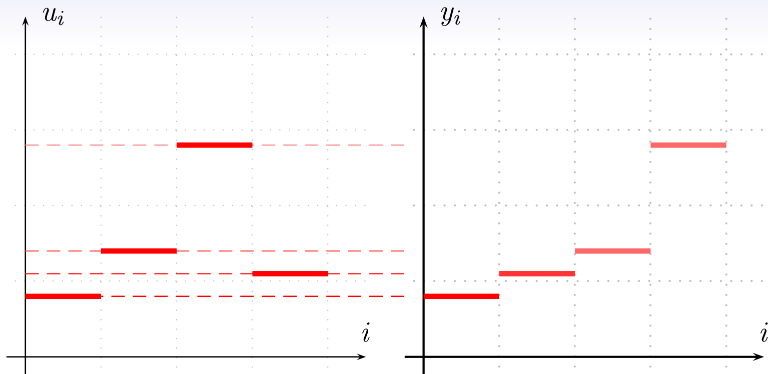

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A sorting constraint

How to ensure that $\langle y_1, \ldots, y_n \rangle$ is the sorted version of the objective vector ?

Constraint Sort

Let $\overrightarrow{x}$ and $\overrightarrow{x'}$ be two vectors of variables of the same size. The constraint $\mathbf{Sort}(\overrightarrow{x}, \overrightarrow{x'})$ holds on the variables of $\overrightarrow{x}$ and $\overrightarrow{x'}$ and only allows the tuples such that $\overrightarrow{x'}$ is the sorted version in incerasing order of $\overrightarrow{x}$.

We use the work from [Bleuzen-Guernalec and Colmerauer, 1997] (a filtering algorithm running in $O(n \log(n))$), and more recently, of [Mehlhorn and Thiel, 2000].

Bleuzen-Guernalec, N. and Colmerauer, A. (1997).
Narrowing a block of sortings in quadratic time.
In *Proc. of CP'97*, pages 2–16, Linz, Austria.

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The algorithm based on the sorting constraint

Algorithm

if $\text{solve}(\mathcal{X}, \mathcal{D}, \mathcal{C}) =$ **"Inconsistent"** **then return** **"Inconsistent"**

$\mathcal{X}' \leftarrow \mathcal{X} \cup \{Y_1, \ldots, Y_n\}$

$\mathcal{D}' \leftarrow \mathcal{D} \cup \{\mathcal{D}_{Y_1}, \ldots, \mathcal{D}_{Y_n}\}$, $\mathcal{D}_{Y_i} = [\![\min_j(\underline{U_j}), \max_j(\overline{U_j})]\!]$

$\mathcal{C}' \leftarrow \mathcal{C} \cup \{\textbf{Sort}(\overrightarrow{U}, \overrightarrow{Y})\}$ **for** $i \leftarrow 1$ **to** n **do**

$\widehat{v}_{(i)} \leftarrow \textbf{maximize}(\mathcal{X}', \mathcal{D}', \mathcal{C}', Y_i)$ $\mathcal{D}_{Y_i} \leftarrow \{\widehat{v}_{(i)}(Y_i)\}$

end

return $\widehat{v}_{(n)\downarrow\mathcal{X}}$

Algorithm 4

Using the saturated subsets

Principle of the algorithm

This algorithm comes from the litterature on fuzzy CSPs [Dubois and Fortemps, 1999].

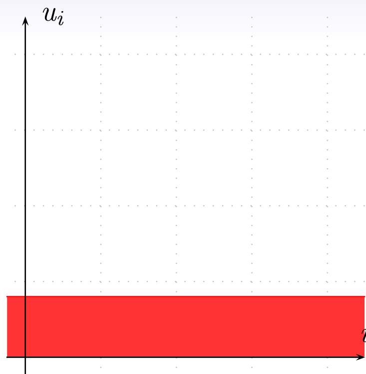
Saturated subset

Let $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ be a constraint networtk, $\overrightarrow{u}$ be a set of objective variables, and α be the maximal admissible value of $\min(\overrightarrow{u})$. A subset $\mathcal{S}$ of variables from $\overrightarrow{u}$ is said **saturated** if there is a solution with all $u_i \in \mathcal{S}$ instantiated to α and all $u_i \notin \mathcal{S}$ instantiated to a strictly greater value than α.

Dubois, D. and Fortemps, P. (1999).
Computing improved optimal solutions to max-min flexible constraint satisfaction problems.
European Journal of Operational Research.

Principle of the algorithm

- At each step of the algorithm, we maximize the current component y_i of the leximin vector (as in the previous algorithms).
- For each cardinality-minimal saturated subset $\mathcal{S}$:
  - all the $u_i \in \mathcal{S}$ are fixed to $\widehat{y_i}$ and are removed from $\overrightarrow{u}$,
  - we restart the procedure on the new constraint network until no variable remains in $\overrightarrow{u}$.
- At the end, we compare all the potential leximin-optimal vectors found.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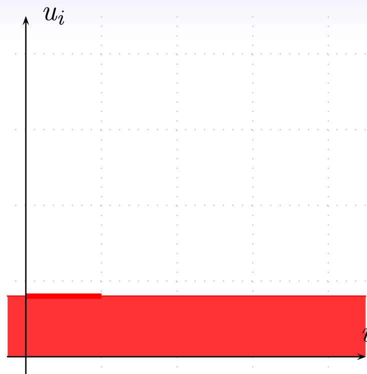
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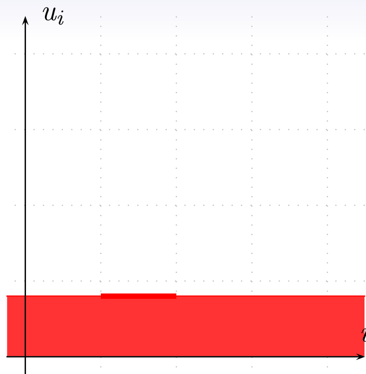

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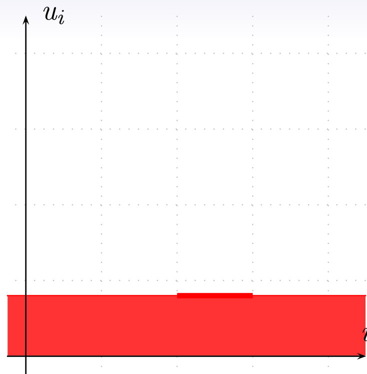
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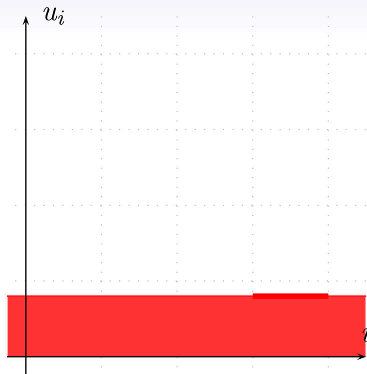
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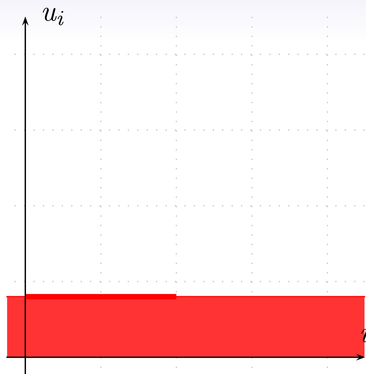
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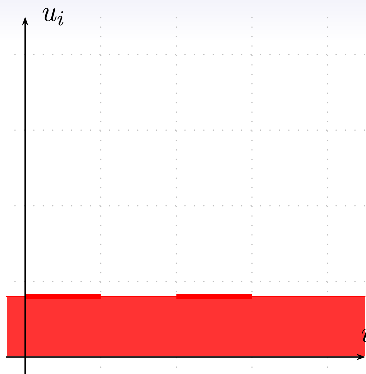
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Major drawback of the algorithm

In the worst case, we can have to explore all the possible subsets of $\overrightarrow{u}$ (if the cardinality-minimal saturated subset is of size n) $\leadsto$ potentially an exponential number of successive resolutions.

Algorithm thus to be used only if we are sure that the cardinality-minimal saturated subsets are of little size.

Outline

1. Formalizing and modeling the problem
   - CSP and constraint programming
   - Fairness in collective decision making
   - Leximin and Constraint Programming
2. Two constraint programming algorithms
   - Using a set of cardinality constraints
   - Using a multiset ordering constraint
   - Sort and conquer
   - Using the saturated subsets
3. Application, tests and results
   - Benchmark and results

Implementation

Implementation of the algorithms:

The two algorithms have been implemented using Choco[F. Laburthe and the OCRE project team, 2000] in Java, and the first one has also been implemented using CPLEX.

Remark

Our specific constraint **AtLeast** can be encoded with a set of linear constraints: $\mathbf{AtLeast}(\{x_1 \geq y, \ldots, x_n \geq y\}, k) \Leftrightarrow$ $\{x_1 + \delta_1 \overline{y} \geq y, \ldots, x_n + \delta_n \overline{y} \geq y, \sum_{i=1}^{n} \delta_i \leq n - k\}$, with $\{\delta_1, \ldots, \delta_n\}$ 0–1 variables.

F. Laburthe and the OCRE project team (2000).
CHOCO: Implementing a CP kernel.
In *Proceedings of TRICKS'2000, Workshop on techniques for implementing Constraint Programming systems*, Singapore.
http://sourceforge.net/projects/choco.

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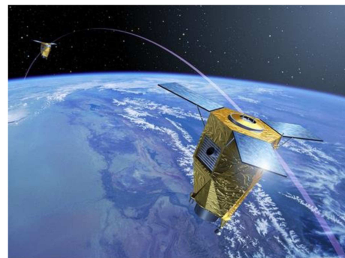
Our real world application

The algorithms have been tested using a (very) simplified model extracted from a real world application:

- Fair share of a constellation of Earth Observation Satellites cofunded by several countries.
- Our simplified model is a multiagent resource allocation with consumption and volume resources.



Instance generator:

We implemented a random instance generator for our simplified problem.

Available online: `http://www.cert.fr/dcsd/THESES/sbouveret/benchmark`

Combinatorial auctions

We have also tested our algorithms on combinatorial auctions instances. A brief description:

- a set of bidders (individuals) and a set of objects;
- each bidder can place some bids, a bid being a set of objects associated with the price the bidder is willing to pay for this set;
- in the traditional formulation, we have to maximize the revenue of the auctioneer (the sum of the prices of non-overlapping selected bids);
- in our formulation, we have to maximize the leximin preorder on the potential utility vectors.

Realistic instances have been generated using CATS (`http://cats.stanford.edu`).

General tendency of the results

- The algorithm based on the **AtLeast** constraint is often the most efficient, followed by the algorithm based on the **Sort** constraint (which has similar running times, except when the number of agents increases).
- The algorithm inspired by the **Multiset ordering** constraint is efficient on the instances of our resource allocation problem with a high number of agents, but completely inefficient with the combinatorial auctions instances.
- The algorithm from [Dubois and Fortemps, 1999] explodes when the number of equal components in the leximin-optimal vector increases.
- Cplex completely outperforms Choco.

Summary

- **Studied problem:** computation of a leximin-optimal allocation of a constraint network.
- **Justification:** the leximin preorder ensures some interesting properties of fairness and efficiency for collective decision making problems.
- **Algorithms:** introduction of two algorithms (the first one being original) based on the CP framework.
- **Implementation:** implementation and testing of the algorithms in Java with Choco (and CPLEX for the first one).

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Current and future work

- **Possible extensions:**
  - More general and softer modeling of fairness (*e.g.* OWA).
  - Applying the algorithms to other practical fields and applications.

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  - Applying the algorithms to other practical fields and applications.

This is the end.

For more information:
sylvain.bouveret@cert.fr
http://www.cert.fr/dcsd/THESES/sbouveret

Random instance generator available online:
http://www.cert.fr/dcsd/THESES/sbouveret/benchmark