Fair division of indivisible goods and compact preference representation: an ordinal approach

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Mara IV Get-Together – June 17-18, 2010

Outline

1. **Fair division**
   - Preferences
   - Envy-freeness
   - Pareto-efficiency
2. **Computing envy-free allocations**
   - Possible envy-freeness
   - Necessary envy-freeness
   - Summary
3. **Beyond separable preferences: Conditional Importance Networks**
   - Language
   - Complexity
   - Fair division

The fair division problem

Given

- a set of indivisible objects $O = \{o_1, \ldots, o_m\}$
- and a set of agents $A = \{1, \ldots, n\}$
- such that each agent has some preferences on the subsets of objects she may receive

Find

- an allocation $\pi : A \to 2^O$
- such that $\pi(i) \cap \pi(j)$ for every $i \neq j$
- satisfying some fairness and efficiency criteria

Preferences

- *cardinal*: agent i has a utility function $u_i : 2^O \to \mathbb{R}$
- *ordinal*: agent i has a preference relation $\succeq_i$ on 2^O

- ordinal preferences are easier to elicit
- ordinality does not require preferences to be interpersonally comparable
- several important criteria need only ordinal preferences

Ordinal preferences

- $\succeq$ weak order (transitive, reflexive and complete relation) on 2^O
- $A \succeq B$: the agent likes A at least as much as B
- strict preference: $A \succ B \Rightarrow A \succeq B$ and not $B \succeq A$
- indifference: $A \sim B \Rightarrow A \succeq B$ and $B \succeq A$

- Preferences over sets of goods are typically *monotonic*: $A \supseteq B \Rightarrow A \succeq B$.

Compact preference representation

- $O = \{o_1, \ldots, o_m\}$
- explicit representation of a preference relation on 2^O: needs exponential space

Possible solutions:

1. restriction on the set of preferences an agent can express (examples: separable preference relations, additive utility functions)
2. compact representation languages

In this talk we consider successively 1 and 2.

Separable ordinal preferences

[Brams et al., 2004, Brams and King, 2005]

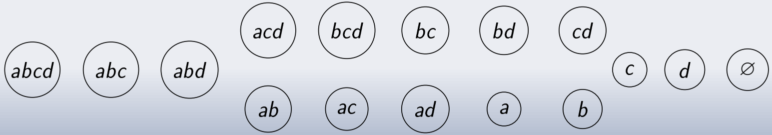
- an agent specifies only a linear order $\rhd$ on single objects
- the partial strict order $\succ_N$ associated with $\rhd$ is the smallest strict order that
  - extends $\rhd$
  - is separable: if $(X \cup Y) \cap Z = \varnothing$ then $X \succ Y$ iff $X \cup Z \succ Y \cup Z$
  - is (strictly) monotonic: if $X \supset Y$ then $X \succ Y$.

Brams, S. J., Edelman, P. H., and Fishburn, P. C. (2004).
Fair division of indivisible items.
Theory and Decision, 5(2):147–180.

Brams, S. J. and King, D. (2005).
Efficient fair division—help the worst off or avoid envy?
Rationality and Society, 17(4):387–421.

Example

- $\mathcal{N} : a \rhd b \rhd c \rhd d$
- Separability
- Monotonicity



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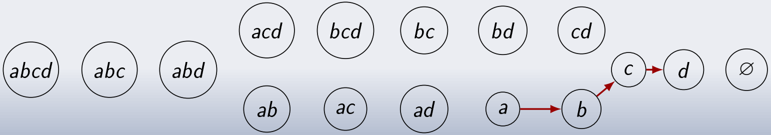

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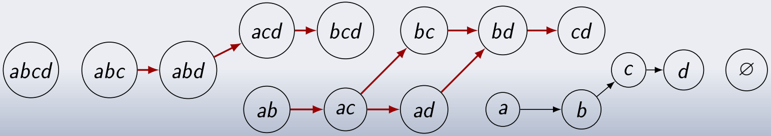

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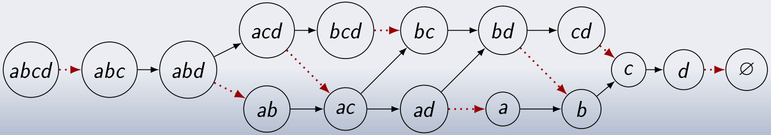

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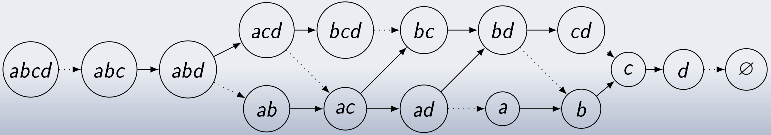

Dominance

An equivalent characterization

$A \succ_{\mathcal{N}} B$ *iff* there exists an injective mapping $g : B \rightarrow A$ such that $g(a) \trianglerighteq_{\mathcal{N}} a$ for all $a \in B$ and $g(a) \triangleright_{\mathcal{N}} a$ for some $a \in B$.

Example: $\mathcal{N} = a \triangleright b \triangleright c \triangleright d \triangleright e \triangleright f$

- $\{ a , c , d \} \succ_{\mathcal{N}} \{ b , c , e \}$
- $\{ a , d , e \}$ and $\{ b , c , f \}$ are incomparable.
- $\{a, c, d\}$ and $\{b, c, e, f\}$ are incomparable.

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Classical envy-freeness

Envy-freeness: classical definition

Given a profile $P = \langle \succ_1, \ldots, \succ_n \rangle$ of **linear** orders: an allocation π is *envy-free* if for all i, j, $\pi(i) \succ_i \pi(j)$

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When $\succ$ is a partial order: envy-freeness becomes a **modal** notion.

Possible and necessary envy-freeness

$P = \langle \succ_1, \ldots, \succ_n \rangle$ collection of strict partial orders; a collection of linear partial orders $P^* = \langle \succ_1^*, \ldots, \succ_n^* \rangle$ is a *completion* of P if for every i, $\succ_i^*$ extends $\succ_i$.

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Possible and necessary Envy-freeness

- π is *possibly envy-free* (PEF) if for some completion P^* of P, π is envy-free with respect to P^*.
- π is *necessarily envy-free* (NEF) if for all completions P^* of P, π is envy-free with respect to P^*.

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A computation-friendly characterization: given $(\succ_1, \ldots, \succ_n)$,

- π is NEF *iff* for all i, j, we have $\pi(i) \succ_i \pi(j)$;
- π is PEF *iff* for all i, j, we have $\pi(j) \not\succ_i \pi(i)$.

Envy-freeness

4 goods, 2 agents

- $\mathcal{N}_1 = a \rhd b \rhd c \rhd d$
- $\mathcal{N}_2 = d \rhd c \rhd b \rhd a$.
- $\pi : 1 \mapsto \{a, d\}; 2 \mapsto \{b, c\}$.
  - $\{b, c\} \not\succ_1 \{a, d\}$ and $\{a, d\} \not\succ_2 \{b, c\}$, therefore π is PEF.
  - π is not NEF
- $\pi' : 1 \mapsto \{a, b\}; 2 \mapsto \{c, d\}$.
  - π is NEF

Pareto-efficiency

Classical Pareto-efficiency

When $\succ$ is linear: π' *Pareto-dominates* π if

- for all i, $\pi'(i) \succeq_i \pi(i)$
- for some i, $\pi'(i) \succ_i \pi(i)$

Possible and necessary Pareto-efficiency

When $\succ$ is a partial strict order:

- π' *possibly dominates* π if π' dominates π in some completion of P ;
- π' *necessarily dominates* π if π' dominates π in all completions of P.

- π is *possibly Pareto-efficient* (PPE) if there exists no allocation π' such that π' necessarily dominates π.
- π' is *necessarily Pareto-efficient* (NPE) if there exists no allocation π' such that π' possibly dominates π.

Envy-freeness and efficiency

Folklore: envy-freeness and Pareto-efficiency cannot always be satisfied simultaneously

Combining envy-freeness and efficiency;

- C1: possible or necessary envy-freeness
- C2: completeness or possible Pareto-efficiency or necessary Pareto-efficiency

Questions:

- under which conditions is it guaranteed that there exists a allocation that satisfies C1 and C2?
- how hard it is to determine whether such an allocation exists?

Complete possibly envy-free allocations

Result

If n agents express their preferences over m goods using SCI-nets and k distinct goods are top-ranked by some agent, then there exists a complete PEF allocation if and only if $m \geq 2n - k$.

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Constructive proof by the following algorithm/protocol:

1. Go through the agents in ascending order, ask them to pick their top-ranked item if it is still available and ask them leave the room if they were able to pick it.
2. Go through the remaining $n - k$ agents in ascending order and ask them to claim their most preferred item from those still available.
3. Go through the remaining agents in descending order and ask them to claim their most preferred item from those still available.

Example

$\mathcal{N}_1$: $a \rhd b \rhd c \rhd d \rhd e \rhd f$ $\mathcal{N}_2$: $a \rhd d \rhd b \rhd c \rhd e \rhd f$
$\mathcal{N}_3$: $b \rhd a \rhd c \rhd d \rhd f \rhd e$ $\mathcal{N}_4$: $b \rhd a \rhd c \rhd e \rhd f \rhd d$

$$(k = 2;\ m = 6 \geq 2n - k)$$

Consider the agents in order $1 > 2 > 3 > 4$:

- *first step*: give a to 1; give b to 3;
- *second step*: give d to 2; give c to 4;
- *third step*: give e to 4; give f to 2.

NPE / PPE-PEF allocations

Result

There exists a PPE-PEF allocation if and only if there exists a complete, PEF allocation.

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Key point of the proof: the previous protocol can be refined into a protocol returning an allocation that is the product of *sincere choices* by the agents in some sequence, and then use a result from [Brams and King, 2005].

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Complexity of the existence of NPE-PEF allocations: *open*

Brams, S. J. and King, D. (2005).
Efficient fair division—help the worst off or avoid envy?
Rationality and Society, 17(4):387–421.

Complete NEF allocations

Two necessary conditions:

- the number m of goods must be a multiple of the n number of agents.
- the top objects must be all distinct

Example (continued):

$$\mathcal{N}_1 \colon a \rhd b \rhd c \rhd d \rhd e \rhd f \qquad \mathcal{N}_2 \colon a \rhd d \rhd b \rhd c \rhd e \rhd f$$
$$\mathcal{N}_3 \colon b \rhd a \rhd c \rhd d \rhd f \rhd e \qquad \mathcal{N}_4 \colon b \rhd a \rhd c \rhd e \rhd f \rhd d$$

- m is not a multiple of n, therefore there is no complete NEF
- if any one of the four agents is removed: idem
- if only agents 1 and 3 are left in: idem
- if only agents 2 and 3 are left in: $\pi(2) = \{a, d, e\}$, $\pi(3) = \{b, c, f\}$ NEF.

Necessary envy-freeness: results

Complete allocation

- deciding whether there exists a complete NEF allocation is NP-complete (even if $m = 2n$).
- the problem falls down in P for two agents

(hardness by reduction from [X3C])

Possible and necessary Pareto-efficiency

- existence of a PPE-NEF allocation: NP-complete
- existence of a NPE-NEF allocation: NP-hard but probably not in NP (Σ_2^p-completeness conjectured).

Complexity results

	complete	PPE	NPE
PEF	P	P	?
NEF	NP-complete	NP-complete (P for 2 agents)	NP-hard (Σ_2^p-completeness conjectured)

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   - Language
   - Complexity
   - Fair division

Beyond separable preferences: CI-nets

Conditional importance statement

$\mathcal{S}^+, \mathcal{S}^- : \mathcal{S}_1 \rhd \mathcal{S}_2$ (with $\mathcal{S}^+$, $\mathcal{S}^-$, $\mathcal{S}_1$ and $\mathcal{S}_2$ pairwise-disjoint).

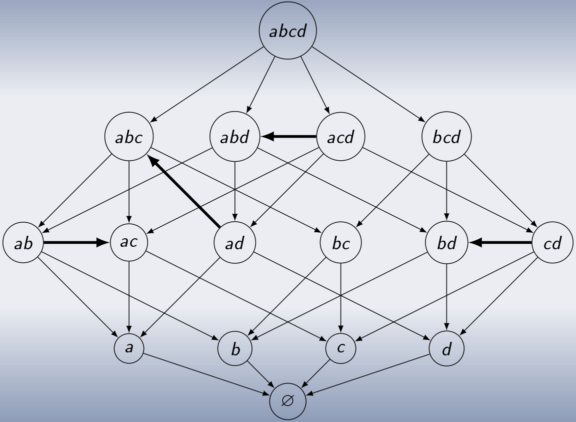
Example: $a\overline{d} : b \rhd ce$ implies for example $ab \succ ace$, $abfg \succ acefg$, …

CI-net

A CI-net on $\mathcal{V}$ is a set $\mathcal{N}$ of conditional importance statements on $\mathcal{V}$.

Semantics

A CI-net of 4 objects $\{a, b, c, d\}$: $\{a : d \triangleright bc, a\overline{d} : b \triangleright c, d : c \triangleright b\}$



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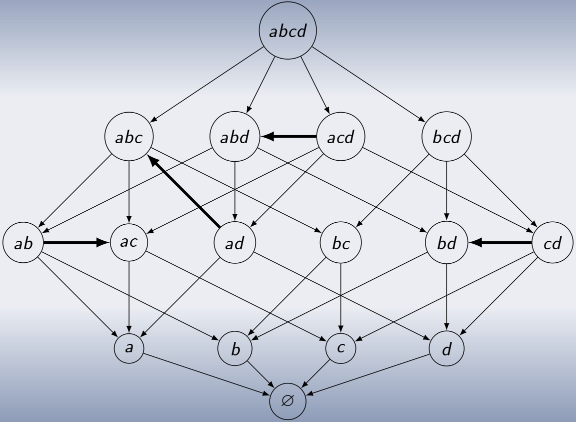

Induced preference relation $\succ_{\mathcal{N}}$: the smallest preference **monotonic** relation compatible with **all CI-statements**.

Worsening flips

Worsening flip

$\mathcal{V}_1 \rightsquigarrow \mathcal{V}_2$ is called a **worsening flip** wrt. $\mathcal{N}$ if:

- either $\mathcal{V}_1 \subseteq \mathcal{V}_2$ (monotonicity flip);
- or they match a CI-statement in $\mathcal{N}$ (CI-flip).

Proposition (dominance)

We have $A \succ_{\mathcal{N}} B$ if and only if there exists a sequence of worsening flips from A to B wrt. $\mathcal{N}$.

Proposition (satisfiability)

A CI-net $\mathcal{N}$ is satisfiable if and only if it does not possess any cycle of worsening flips.

Expressivity

Proposition

CI-nets can express all strict monotonic preference relations on $2^{\mathcal{V}}$.

Proof sketch: for every (X, Y) such that $X \succ Y$ and $X \not\supset Y$, add the CI-statement $(X \cap Y), (\overline{X \cup Y}) : X \setminus Y \rhd Y \setminus X$.

Proposition

Full expressivity is lost as soon as:

(i) we do not allow positive preconditions;

(ii) we do not allow negative preconditions;

(iii) the cardinality of compared sets is bounded by a fixed integer.

Dominance

[DOMINANCE]

Input: A (satisfiable) CI-net $\mathcal{N}$, two bundles X and Y.

Question: $X \succ_{\mathcal{N}} Y$?

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Some bad news...

Proposition

[DOMINANCE] in satisfiable CI-nets is **PSPACE**-complete, even under any of these restrictions:

1. every CI-statement bears on singletons and has no negative preconditions;
2. every CI-statement bears on singletons and has no positive preconditions;
3. every CI-statement is precondition-free.

Dominance

[DOMINANCE]

Input: A (satisfiable) CI-net $\mathcal{N}$, two bundles X and Y.
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Back to part 1 of the talk...
SCI-nets: precondition-free, singleton-comparing CI-statements.
Example: $\{a \rhd c, b \rhd c, e \rhd d\}$.

Proposition

[DOMINANCE] in satisfiable SCI-nets is in P.

Satisfiability

[SATISFIABILITY]

Input: A CI-net $\mathcal{N}$.
Question: Is $\mathcal{N}$ satisfiable ?

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Some good news...

- [SATISFIABILITY] for SCI-nets is in P.
- Two **sufficient** conditions for satisfiability: based on **acyclicity**.

CI-nets and fair division

Example

- **Objects**: $\mathcal{V} = \{a, b, c\}$.
- **Agents**:
  - $\mathcal{N}_1 = \{b : c \rhd a, \overline{b} : a \rhd c\}$;
  - $\mathcal{N}_2 = \{c \rhd a, a \rhd b\}$

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- $\langle 1 : a, 2 : bc \rangle$ is not possibly envy-free.
- $\langle 1 : b, 2 : ac \rangle$ is possibly envy-free but not necessarily envy-free.

However: all existence problems are now PSPACE-complete!

$\Rightarrow$ other tractable fragments (than SCI-nets)

$\Rightarrow$ approximate dominance relation

Conclusion

Fair division with incomplete ordinal preferences:

- separable and monotone ordinal preferences (SCI-nets);
- modal Pareto-efficiency and Envy-freeness;
- extension to non-separable preferences $\rightsquigarrow$ CI-nets.

Results:

- **SCI-nets**: fair division not tractable (NP-hard) in general;
- **CI-nets**: even dominance is far beyond untractability (PSPACE-complete).

Solutions ?

$\Rightarrow$ other fairness criteria (than envy-freeness);

$\Rightarrow$ other tractable fragments (than SCI-nets);

$\Rightarrow$ approximate dominance relation.