Mon partage sera-t-il conflictuel ?

Une échelle de propriétés pour la caractérisation d'instances de partage de biens indivisibles

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7èmes journées francophones Modèles Formels de l'Interaction

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Fair division of indivisible goods. . .

We have:

- a finite set of **objects** $\mathcal{O} = \{1, \ldots, m\}$
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- an allocation $\overrightarrow{\pi} : \mathcal{A} \to 2^{\mathcal{O}}$
- such that $\pi_i \cap \pi_j = \emptyset$ if $i \neq j$ (preemption),
- $\bigcup_{i \in \mathcal{A}} \pi_i = \mathcal{O}$ (no free-disposal),
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Plenty of real-world applications: course allocation, operation of Earth observing satellites, . . .

Centralized allocation

A classical way to solve the problem:

- Ask each agent i to give a score (weight, utility...) $w_i(o)$ to each object o
- Consider all the agents have **additive** preferences

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- Find an allocation $\overrightarrow{\pi}$ that:

1. maximizes the collective utility defined by a **collective utility function**,
   e.g. $uc(\overrightarrow{\pi}) = \min_{i \in \mathcal{A}} u(\pi_i)$ – egalitarian solution
   [Bansal and Sviridenko, 2006]
2. or satisfies a given **fairness criterion**,
   e.g. $u_i(\pi_i) \geq u_i(\pi_j)$ for all agents i, j – envy-freeness
   [Lipton et al., 2004].

Bansal, N. and Sviridenko, M. (2006).
The Santa Claus problem.
In *Proceedings of STOC'06*. ACM.

Lipton, R., Markakis, E., Mossel, E., and Saberi, A. (2004).
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$\overrightarrow{\pi}$ is **not** envy-free (agent 1 envies agent 2)

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1. such an allocation does not always exist
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Idea: consider several fairness properties, and try to satisfy the most demanding one.
In this work we consider five such properties.

Outline

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Known facts:

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- Deciding whether an instance (agents, objects, preferences) has an envy-free allocation is hard – **NP**-complete [Lipton et al., 2004].

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Proportional fair share (PFS):

- Initially defined by Steinhaus [Steinhaus, 1948] for continuous fair division (*cake-cutting*)
- **Idea:** each agent is "entitled" to at least the nth of the entire resource

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Proportional fair share

The **proportional fair share** of an agent i is equal to:

$$u_i^{\mathrm{PFS}} \stackrel{def}{=} \frac{u_i(\mathcal{O})}{n} = \sum_{o \in \mathcal{O}} \frac{w_i(o)}{n}$$

An allocation $\overrightarrow{\pi}$ satisfies **(proportional) fair share** if every agent gets at least her fair share.

Proportional fair share: facts

Easy or known facts:

- Deciding whether an allocation satisfies proportional fair share (PFS) is easy (linear time).
- For a given instance, there may be no allocation satisfying PFS
  → *e.g.* 2 agents, 1 object
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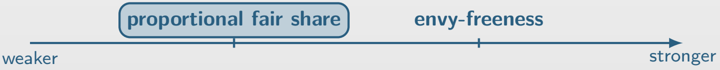

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- **Idea:** in the **cake-cutting** case, PFS = the best share an agent can hopefully get for sure in a *"I cut, you choose (I choose last)"* game.
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An allocation $\overrightarrow{\pi}$ satisfies **max-min fair share** (MFS) if every agent gets at least her max-min fair share.

Max-min fair share: examples

Example: 3 objects $\{1, 2, 3\}$, 2 agents $\{1, 2\}$.

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$u_1^{\mathrm{MFS}} = u_2^{\mathrm{MFS}} = 0 \rightarrow$ every allocation satisfies MFS!

Not very satisfactory, but can we do much better?

Max-min fair share: properties

Facts:

- Computing u_i^{MFS} for a given agent is hard $\rightarrow$ **NP**-complete [PARTITION]
- Hence, deciding whether an allocation satisfies MFS is also hard.
- $\overrightarrow{\pi}$ satisfies PFS $\Rightarrow$ $\overrightarrow{\pi}$ satisfies MFS.

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- mFS $=$ the worst share an agent can get in a *"Someone cuts, I choose first"* game.
- In the **cake-cutting** case, same as PFS.

Min-max fair share: properties

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- Hence, deciding whether an allocation satisfies mFS is also hard.
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- $\overrightarrow{\pi}$ is envy-free $\Rightarrow$ $\overrightarrow{\pi}$ satisfies mFS.

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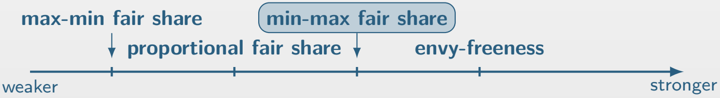

Competitive Equilibrium from Equal Incomes (CEEI)

- Set one price $p_o \leq$ £1 for each object o.
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- Let $\pi_i^\star$ be (among) the best share(s) agent i can buy with her £1.
- If $(\pi_1^\star, \ldots, \pi_n^\star)$ is a valid allocation, it forms, together with $\overrightarrow{p}$, a **CEEI**.

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- Classical notion in economics [Moulin, 1995]
- Not so much studied in computer science (except [Othman et al., 2010])

Moulin, H. (1995).
Cooperative Microeconomics, A Game-Theoretic Introduction.
Prentice Hall.

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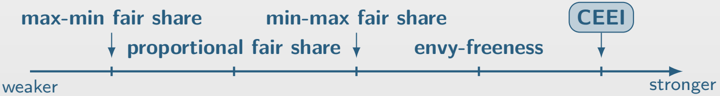

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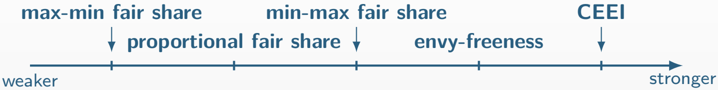

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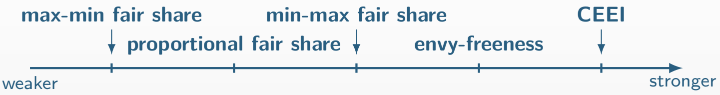

1. For all allocation $\overrightarrow{\pi}$:

$$(\overrightarrow{\pi} \models \mathrm{CEEI}) \Rightarrow (\overrightarrow{\pi} \models \mathrm{EF}) \Rightarrow (\overrightarrow{\pi} \models \mathrm{mFS}) \Rightarrow (\overrightarrow{\pi} \models \mathrm{PFS}) \Rightarrow (\overrightarrow{\pi} \models \mathrm{MFS})$$

$\rightarrow$ the highest property $\overrightarrow{\pi}$ satisfies, the most satisfactory it is.

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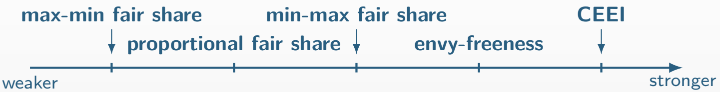

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$$\mathcal{I}_{|\mathrm{CEEI}} \subset \mathcal{I}_{|\mathrm{EF}} \subset \mathcal{I}_{|\mathrm{mFS}} \subset \mathcal{I}_{|\mathrm{PFS}} \subset \mathcal{I}_{|\mathrm{MFS}} (= \mathcal{I}?)$$

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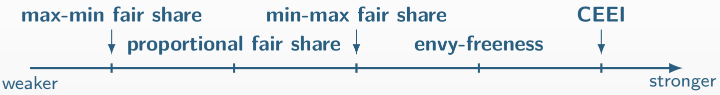

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Two extreme examples:

- 2 agents, 1 object $\rightarrow$ only in $\mathcal{I}_{|\text{MFS}}$
- 2 agents, 2 objects, with

	1	2
agent 1	1000	0
agent 2	0	1000

$\rightarrow$ in $\mathcal{I}_{|\text{CEEI}}$ (with *e.g.* $\overrightarrow{p} = \langle 1, 1 \rangle$).

Additional properties

1. **Strict inclusions** $\mathcal{I}_{|\mathrm{CEEI}} \subset \mathcal{I}_{|\mathrm{EF}} \subset \mathcal{I}_{|\mathrm{mFS}} \subset \mathcal{I}_{|\mathrm{PFS}} \subset \mathcal{I}_{|\mathrm{MFS}} (= \mathcal{I}?)$

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Are these inclusions strict? Yes, they are, and we can prove it!

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- **Envy-freeness:** question studied in [Brams and King, 2005]
- **Max-min fair share:** egalitarian optimal allocations **almost always satisfy** max-min fair share.

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3. **Interpersonal comparison**
   - Egalitarianism requires the preferences to be **comparable**:
     - either expressed on a same scale (*e.g.* money)...
     - ...or normalized (*e.g.* Kalai-Smorodinsky)
   - The five fairness criteria introduced do not (**independence of the individual utility scales**).

$\rightarrow$ This is a very appealing property.

Outline

MFS and k-additive preferences

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Conjecture

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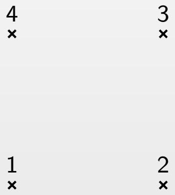

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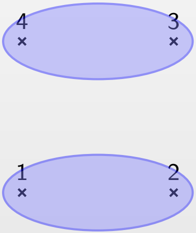

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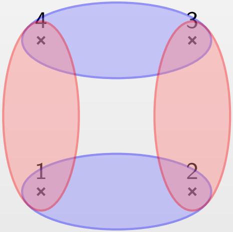

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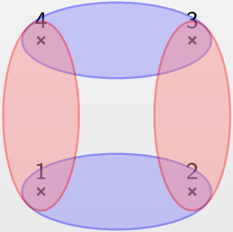

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Worse... Deciding whether there exists one is **NP**-complete [Partition].

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A possible approach to fairness in multiagent resource allocation problems:

1. Determine the highest satisfiable criterion.
2. Find an allocation that satisfies this criterion.
3. Explain to the upset agents that we cannot do much better.

Future directions

- Close the **conjecture** and missing complexity results.
- Develop efficient **algorithms** (possibly in conjunction with approximation of fairness criteria)
- **Experiments**: Build a cartography of resource allocation problems.
- Extend the results to **more expressive preference languages**.

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- The five criteria do not require interpersonal comparison of utilities.
- Moreover: Four of them are **purely ordinal** (PFS is not)
- Do the results extend to (separable) **ordinal preferences** ?