

Picking Sequences for Resource Allocation

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My coauthors...

The papers presented in this talk are the result of a joint work with:

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- Hugo Gilbert
- Guillaume Méroué
- Jérôme Lang



Fair division of indivisible goods...



Fair division of indivisible goods...











Fair division of indivisible goods...





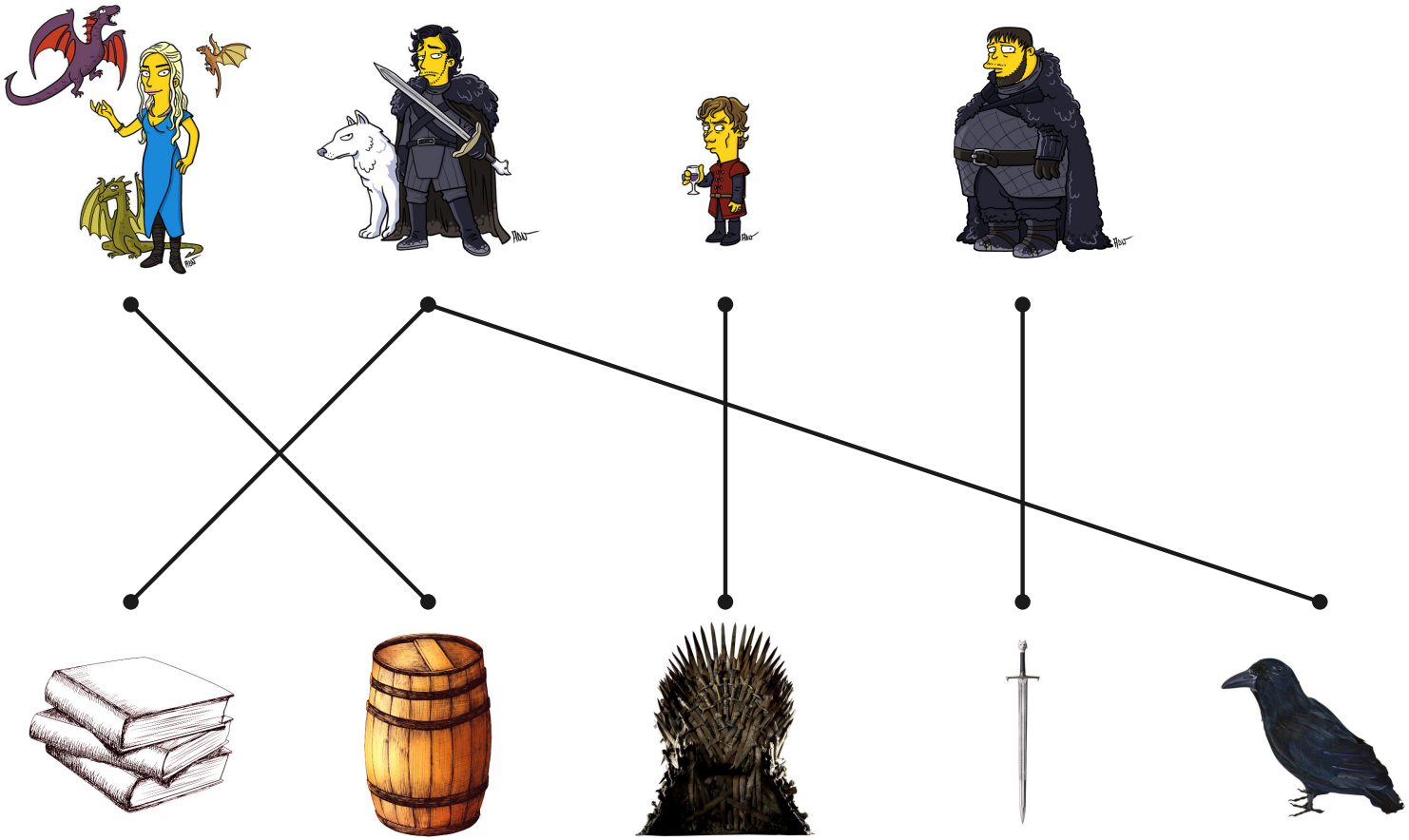









Fair division of indivisible goods...





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1. Ask the agents to give their preferences and use a (centralized) collective decision making procedure.



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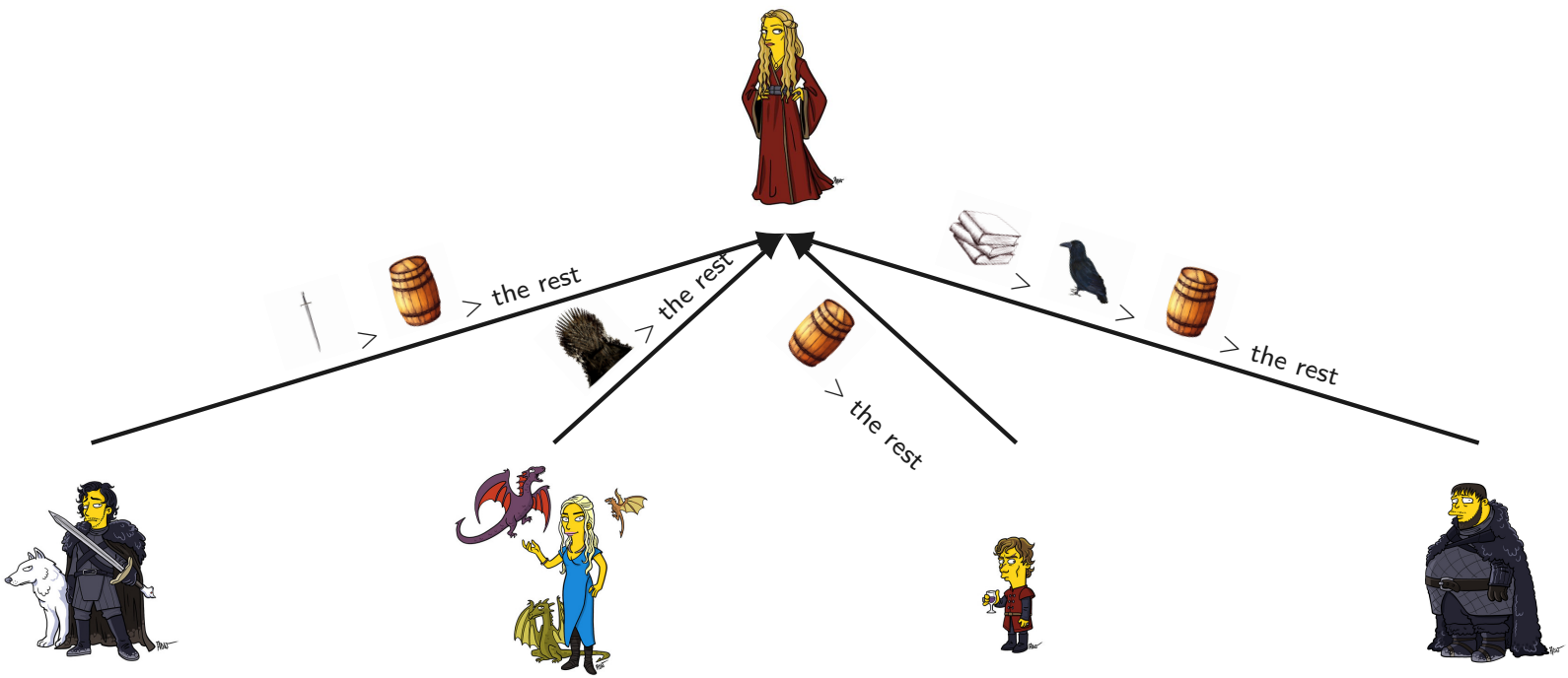






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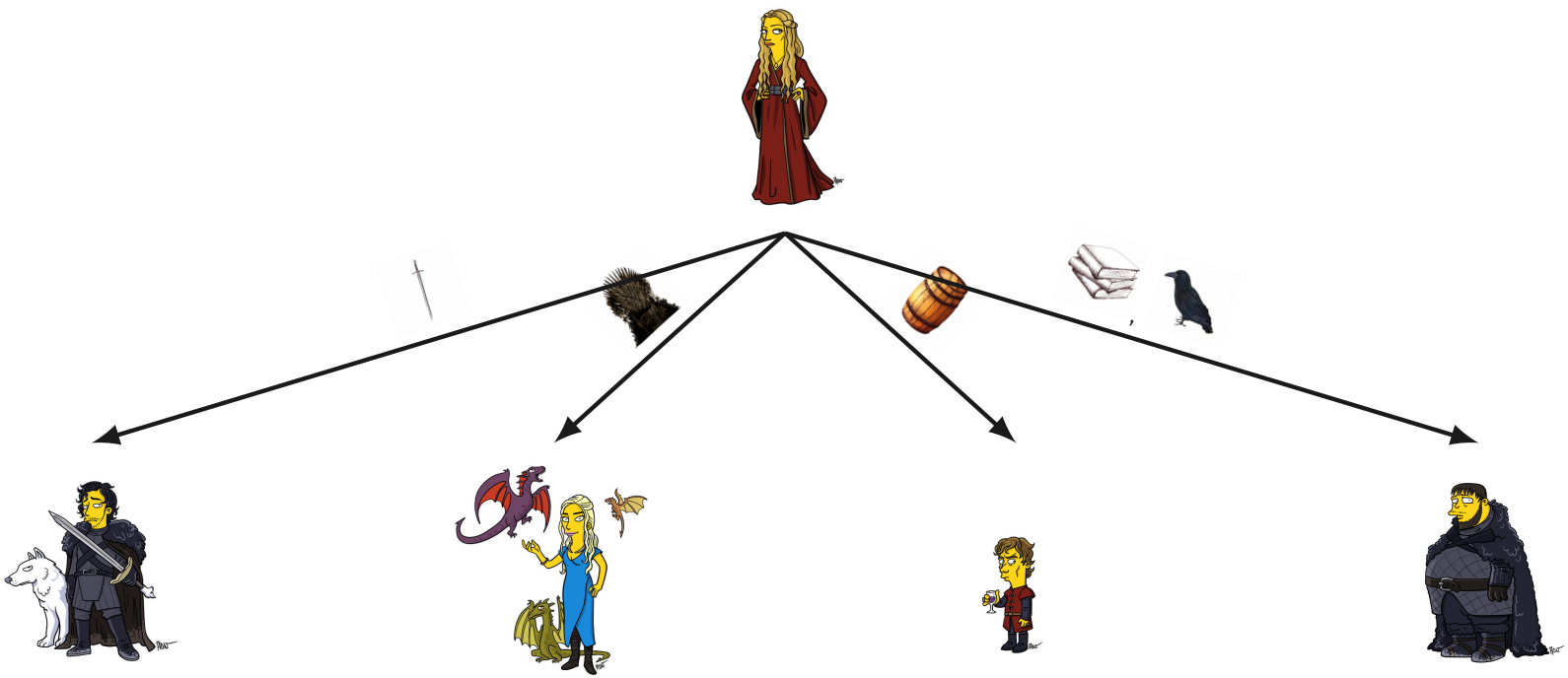
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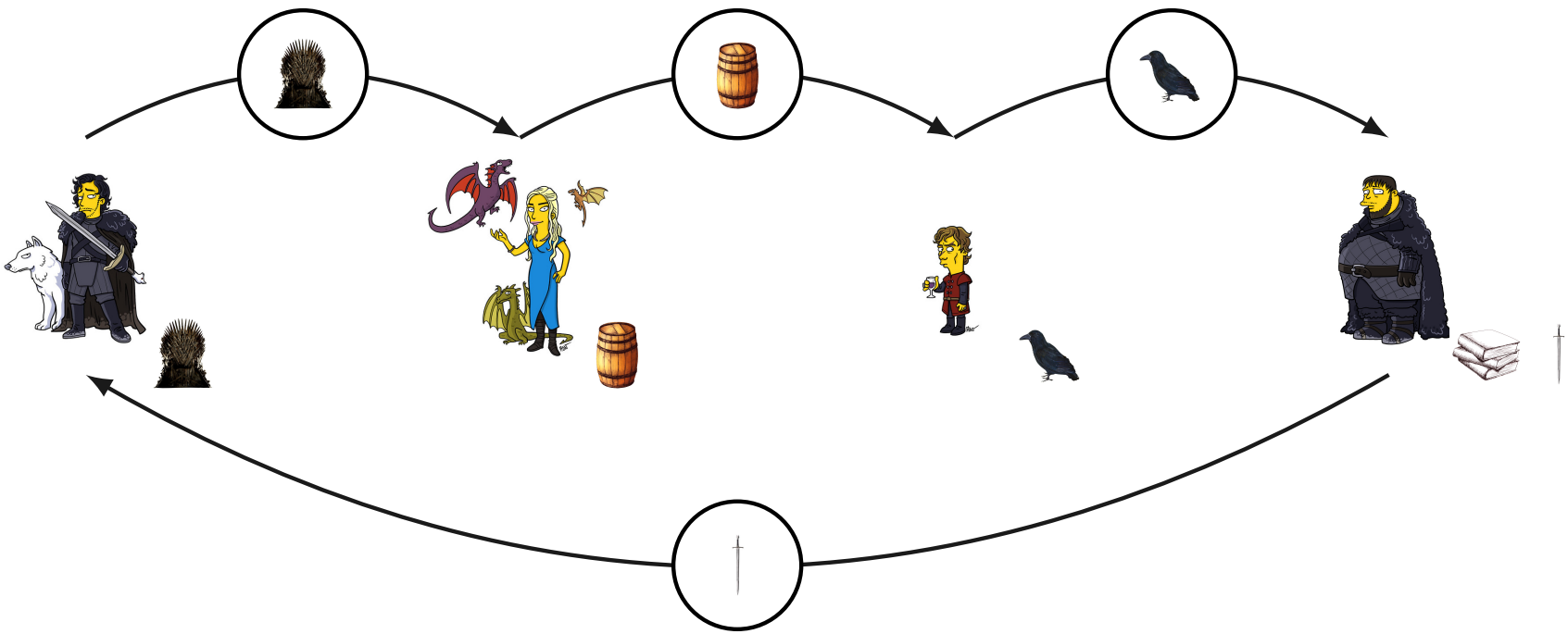







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In this presentation, we will focus on **picking sequences**



More formally...

Ask the individuals to pick in turn their most preferred object among the remaining ones, according to some **predefined sequence**.

Example

3 individuals A, B, C, 6 items, sequence $ABCCBA \rightarrow A$ chooses first (and takes her preferred item), then B, then C, then C again...



Sequences

- Natural and simple protocol
- Used in practice
- Preference elicitation-free

- Board games
- Draft mechanisms (sport)
- Course allocation (Harvard Business School)
- ...





The optimal sequence problem

We “feel” that *ABCCBA* is **fairer** than *AABBCC*...



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Given a number of agents and a number of objects (+ some additional assumptions), what is the **fairest** sequence?



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The optimal sequence problem

Given a number of agents and a number of objects (+ some additional assumptions), what is the **fairest** sequence?

Today, we will mostly focus on the optimal sequence problem (+ touch upon some other notions like efficiency and strategyproofness)

The optimal sequence problem

What is the fairest sequence?



Agents, objects, preferences...

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A central authority (CA) must find a **policy** (a sequence of agents) $\pi : \{1, \ldots, p\} \to \{A, B, \ldots, x\}$
- The central authority assumes that each agent i has a (private) **ranking** $\succ_i$ over $\mathcal{O}$ (ex: $6 \succ 1 \succ 4 \succ 5 \succ 2 \succ 3$)



Example

Example

5 objects, 3 agents, $\pi = ABCCB\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$



Example

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- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
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k	0
$s(A)_k^\pi$	$\emptyset$
$s(B)_k^\pi$	$\emptyset$
$s(C)_k^\pi$	$\emptyset$
$\mathcal{O}_k^\pi$	$\emptyset$



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k	0	1
$s(A)_k^\pi$	$\emptyset$	1
$s(B)_k^\pi$	$\emptyset$	$\emptyset$
$s(C)_k^\pi$	$\emptyset$	$\emptyset$
$\mathcal{O}_k^\pi$	$\emptyset$	1



Example

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5 objects, 3 agents, $\pi = A\mathbf{B}CCB\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
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k	0	1	2
$s(A)_k^\pi$	$\emptyset$	1	1
$s(B)_k^\pi$	$\emptyset$	$\emptyset$	4
$s(C)_k^\pi$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathcal{O}_k^\pi$	$\emptyset$	1	14



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k	0	1	2	3
$s(A)_k^\pi$	$\emptyset$	1	1	1
$s(B)_k^\pi$	$\emptyset$	$\emptyset$	4	4
$s(C)_k^\pi$	$\emptyset$	$\emptyset$	$\emptyset$	3
$\mathcal{O}_k^\pi$	$\emptyset$	1	14	143



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k	0	1	2	3	4
$s(A)_k^\pi$	$\emptyset$	1	1	1	1
$s(B)_k^\pi$	$\emptyset$	$\emptyset$	4	4	4
$s(C)_k^\pi$	$\emptyset$	$\emptyset$	$\emptyset$	3	35
$\mathcal{O}_k^\pi$	$\emptyset$	1	14	143	1435



Example

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5 objects, 3 agents, $\pi = ABCC\mathbf{B}\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

k	0	1	2	3	4	5
$s(A)^\pi_k$	$\emptyset$	1	1	1	1	1
$s(C)^\pi_k$	$\emptyset$	$\emptyset$	4	4	4	42
$s(C)^\pi_k$	$\emptyset$	$\emptyset$	$\emptyset$	3	35	35
$\mathcal{O}^\pi_k$	$\emptyset$	1	14	143	1435	14352



Scoring functions

We only have rankings over objects...

$\rightarrow$ *How to compare two allocations ?*



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Two natural assumptions:

1. **Scoring:** We have a common **scoring function** $g : \{1, \ldots, p\} \mapsto \mathbb{N}$ mapping each rank to a utility.
2. **Additivity:** These utilities are **additive**.



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With π, agent A gets 1, agent B gets 24, agent C gets 35



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With π, agent A gets 1, agent B gets 24, agent C gets 35

- **Borda:** $u_A(\pi) = 5; u_B(\pi) = 5 + 4 = 9; u_C(\pi) = 4 + 3 = 7.$
- **lexicographic:** $u_A(\pi) = 16; u_B(\pi) = 24; u_C(\pi) = 12.$
- **QI:** $u_A(\pi) = 1 + 4\varepsilon; u_B(\pi) = 2 + 7\varepsilon; u_C(\pi) = 2 + 5\varepsilon.$



Social welfare

We use a **collective utility function** to aggregate the individual utilities.

Two well-known functions:

- **utilitarian:** $F(u_A, \ldots, u_x) = \sum_{i \in \mathcal{N}} u_i$;
- **egalitarian:** $F(u_A, \ldots, u_x) = \min_{i \in \mathcal{N}} u_i$;
- (We will also speak about **Nash** in the second part: $F(u_A, \ldots, u_x) = \prod_{i \in \mathcal{N}} u_i$).



Uncertainty

The procedure is elicitation-free. . .

$\rightarrow$ *Which information can we use to find the best sequence ?*



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The CA has a **prior probability** on the preference profile:

- **Impartial Culture** (IC): each profile $R = \langle \succ_A, \ldots, \succ_x \rangle$ is equally probable
- **Full correlation** (FC): all the agents have the same ranking ($R = \langle \succ, \ldots, \succ \rangle$)



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Expected individual and collective utilities:

$$EU^{\Psi}(i, \pi) = \mathbb{E}_{R \sim \Psi}[u^R(i, \pi)] = \sum_{R \in Prof(\mathcal{N}, \mathcal{O})} Pr_{\Psi}(R) \times u_i^R(\pi, R).$$

$$ESW_F^{\Psi}(\pi) = F(EU^{\Psi}(1, \pi), \ldots, EU^{\Psi}(n, \pi)).$$



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5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, IC.

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What is agent C's expected utility with this sequence ?

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$$EU^{IC}(3, \pi) = 0.5 + \frac{1}{\binom{5}{2}} \times (4 + 2)$$



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What is agent C's expected utility with this sequence ?

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$$EU^{IC}(3, \pi) = 0.5 + 0.6 + \frac{1}{\binom{5}{2}} \times (5 + 2)$$



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What is agent C's expected utility with this sequence ?

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$$EU^{IC}(3, \pi) = 0.5 + 0.6 + 0.6 + \frac{2}{\binom{5}{2}} \times (4 + 3)$$



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$$EU^{IC}(3, \pi) = 0.5 + 0.6 + 0.6 + 1.4 + \frac{2}{\binom{5}{2}} \times (5 + 3)$$



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What is agent C's expected utility with this sequence ?

C's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$EU^{IC}(3, \pi) = 0.5 + 0.6 + 0.6 + 1.4 + 1.6 + 2.7 = \mathbf{7.5}$$



Summary

- **Instance:**
  - a number of agents n
  - a number of objects p
  - a scoring function g
  - a prior (*i.e* a correlation assumption) $\Psi \in \{FC, IC\}$
  - a collective utility function F
- **Question:**
  - *What is the policy π maximizing $ESW_F^{\Psi}(\pi)$, under correlation profile Ψ?*



Some general results

1. **Full correlation**



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Utilitarian CUF (sum)

All policies have the same expected value!



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All policies have the same expected value!

Egalitarian CUF (min)

Sequential allocation is **NP**-complete (actually, pseudo-polynomial).

(Reduction from [Partition])



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All policies have the same expected value!

Egalitarian CUF (min)

Sequential allocation is **NP**-complete (actually, pseudo-polynomial).

(Reduction from [Partition])

What about...

- ...lexicographic scoring ?
- ...quasi-indifference scoring ?
- ...Borda scoring ?



Lexicographic scoring

$\succ_i$	6	$\succ$	1	$\succ$	4	$\succ$	5	$\succ$	2	$\succ$	3
lexicographic	32	$\gg$	16	$\gg$	8	$\gg$	4	$\gg$	2	$\gg$	1

Egalitarian CUF (min)

Optimal policies: $\sigma(A)\sigma(B)\ldots\sigma(x)\sigma(x)^{p-n}$ (where σ is a permutation of $\{A, B, \ldots, x\}$)



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Egalitarian CUF (min)

Optimal policies: $\sigma(A)\sigma(B)\ldots\sigma(x)\sigma(x)^{p-n}$ (where σ is a permutation of $\{A, B, \ldots, x\}$)

Example: $\pi = ABCCCC$

- $EU^{FC}(1, \pi) = 32$
- $EU^{FC}(2, \pi) = 16$
- $EU^{FC}(3, \pi) = 8 + 4 + 2 + 1 = 15$



Borda scoring

$\succ_i$	6	1	4	5	2	3
Borda	6	5	4	3	2	1



Borda scoring

$\succ_i$	6	1	4	5	2	3
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Egalitarian CUF (min)

This is **polynomial** in p.

Hint: The number of possible utility values is bounded for an agent ($p(p+1)/2$) $\rightarrow$ use a dynamic programming algorithm.



QI scoring

$\succ_i$	6	1	4	5	2	3
Quasi-Indifference	$1+5\varepsilon$	$1+4\varepsilon$	$1+3\varepsilon$	$1+2\varepsilon$	$1+\varepsilon$	1

Egalitarian CUF (min)

Comes down to solving the Borda case!



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Egalitarian CUF (min)

Comes down to solving the Borda case!

Intuition:

- let $m = \lfloor \frac{p}{n} \rfloor$ and $q = p - nm$
- Optimal policies: $\pi = \underbrace{AABB}_{n-q \text{ agents}} \overbrace{CCCDDD}^{q \text{ agents}}$ and $\pi' = \underbrace{ABBA}_{n-q \text{ agents}} \overbrace{CCCDDD}^{q \text{ agents}}$
- The q last agents are OK $\rightarrow u \geq m+1$
- The $n-q$ first agents: $u = m + x \cdot \varepsilon$ ($x \rightarrow$ Borda)



A complex problem...

2. Impartial Culture



A complex problem...

2. **Impartial Culture**

Conjecture (2011)

Computing the expected utility of a sequence is **NP**-complete.
Computing the **optimal** sequence probably harder.



Results

Computing the expected utility of a sequence is **NP**-complete.



Results

Computing the expected utility of a sequence is ~~**NP-complete**~~ **polynomial** [Kalinowski et al., 2013].



Kalinowski, T., Narodytska, N., and Walsh, T. (2013).
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Results

Computing the expected utility of a sequence is ~~**NP-complete**~~ **polynomial** [Kalinowski et al., 2013].

Computing the **optimal** sequence probably harder.

- the **alternating policy** (*ABABABAB*...) is optimal for Borda, utilitarian social welfare
- complexity unknown for other social welfare and scoring functions (**NP**-hardness conjectured)

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Some examples

Assumptions: Impartial Culture, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4		
5		
6		
8		
10		



Some examples

Assumptions: Impartial Culture, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4	ABBA	
5		
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Some examples

Assumptions: Impartial Culture, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4	ABBA	ABCC
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p	$n = 2$	$n = 3$
4	ABBA	ABCC
5	AABBB	ABCCB
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Some examples

Assumptions: Impartial Culture, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4	ABBA	ABCC
5	AABBB	ABCCB
6	ABABBA	ABCCBA
8	ABBABAAB	AACCBBCB
10	ABBAABABBA	ABCABBCACC

A small digression about strategical issues (manipulation)

Is the protocol strategy-proof?



Manipulation?

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A central authority (CA) has chosen a **policy** π and will execute it
- The agents have their own private preferences $\rightarrow$ **picking strategy**.



Manipulation?

Example

2 agents, 4 objects:

- A: $1 \succ 2 \succ 3 \succ 4$
- B: $2 \succ 3 \succ 4 \succ 1$

Sequence $\pi = ABBA \rightarrow \{14|23\}$.



Manipulation?

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What if A knows B's preferences and acts maliciously?

She can manipulate by picking 2 instead of 1 at first step $\rightarrow \{12|34\}$.



More formally

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A **policy** π
- The agents have their own private preferences (which may or may not be additive) and use them for their **picking strategy**.



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The cheating agent (A) knows:

- the sequence
- her own (general) **picking strategy**
- the others' **picking strategy** (assumed to be **simple** and **deterministic** – as if each agent had an underlying linear order over the objects)



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- The agents have their own private preferences (which may or may not be additive) and use them for their **picking strategy**.

The cheating agent (A) knows:

- the sequence
- her own (general) **picking strategy**
- the others' **picking strategy** (assumed to be **simple** and **deterministic** – as if each agent had an underlying linear order over the objects)

She wants:

- to get the best bundle she can get.



More formally

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A **policy** π
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She wants:

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Her only possible cheating actions:

- choose at given steps **not** to pick her preferred objects.



Bad news...

(Folk?) theorem

The only strategyproof picking sequences are those made of contiguous blocks of agents (e.g. $A...AB...BC...C$).



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The only strategyproof picking sequences are those made of contiguous blocks of agents (e.g. $A...AB...BC...C$).

Two possible actions to prevent manipulation as a mechanism designer:

- Impose strategyproofness
- Use complexity as a barrier to manipulation

Strategyproof picking sequences

What can we do if we impose strategyproofness in picking sequences?



Non interleaving sequences

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The only strategyproof picking sequences are **non-interleaving** sequences (e.g. $A...AB...BC...C$).



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- A non-interleaving sequence is defined by a vector $(k_1, \ldots, k_n)$ such that $\sum_{i=1}^{n} k_i = m$
- Can we still ensure **fairness** in non-interleaving sequences?



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$\rightarrow$ We can compensate late arrival by higher number of goods picked.



Finding the best sequence

- For regular (general) sequences, finding the fairest sequence was complex in most (interesting) cases
- What about **non-interleaving** ones?



Finding the best sequence

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- What about **non-interleaving** ones?

We use the same model as for general sequences (based on priors on preference orderings), except that we add:

- The **Mallows** model, parameterized by a ranking μ and a dispersion ϕ:
  - probability of ranking r: proportional to $\phi^{d_{\mathrm{KT}}(r,\mu)}$
- The **Placket-Luce** model, parameterized by a vector $\overrightarrow{\nu} = (\nu_1, \ldots, \nu_m)$ (intuitively, ν_i is item o_i's social value)
  - probability of ranking $r = o_{i_1} \succ \cdots \succ o_{i_m}$:

$$\prod_{j=1}^{m} \frac{\nu_{i_j}}{\sum_{l=j}^{m} \nu_{i_l}}$$



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NB: FC and IC are two subcases of these general models.



Summary of the model

Let:

- Ψ be a prior ($\text{Mll}_{\mu,\phi}$, PL_{ν}, IC or FC)
- F be a collective utility function (ut, eg, Nash)

The OptSD-Ψ-F problem

- **Instance:**
  - a number of agents n
  - a number of objects m
  - a scoring function g
- **Find:**
  - *A vector* $\overrightarrow{k}$ *with* $\sum_{i=1}^{n} k_i = m$ *maximizing* $SW_{\Psi}^{F}(\overrightarrow{k})$



Note (small digression)

You may have already heard of non-interleaving picking sequences, that have some similarities with **sequential/serial dictatorship**



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You may have already heard of non-interleaving picking sequences, that have some similarities with **sequential/serial dictatorship**

- Serial dictatorship:
  - a fixed sequence of agents
  - the 1st one takes her best bundle
  - then the 2nd one takes her best bundle among the remaining items
  - and so on...
- Sequential dictatorship: same, but the sequence depends on the preference profile [Pápai, 2001]
- Sequential quota choice function: same as non-interleaving picking sequences [Pápai, 2000]

Pápai, S. (2000).

original papers : Strategyproof multiple assignment using quotas.
Review of Economic Design, 5(1):91–105.

Pápai, S. (2001).

Strategyproof and Nonbossy Multiple Assignments.
Journal of Public Economic Theory, 3(3):257–271.



Computing the best expected SW

What is the best sequence $\overrightarrow{k} = (k_1, \ldots, k_n)$*?*



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Note: Here, v_1 only depends on k_1 and is easy to compute (the utility of the k_1 best objects)



Computing the best ESW

- Now let us switch to Agent $i > 1$



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Note: Here, v_i depends on k_i but **also on** $k_1, \ldots, k_{i-1}$, but...



Computation of v

...For a few probability distributions Ψ:

Key property

v_i only depends on:

- # of items received by $i \rightarrow k_i$
- # of items picked before $\rightarrow t = \sum_{j=1}^{i-1} k_j$.



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- If $\Psi = FC$: $v(k,t) = \sum_{i=t+1}^{t+k} g(i)$ (can be computed in time $O(m^2)$)
- If $\Psi = IC$, $v(k,t)$ can be computed in time $O(m^3)$ by dynamic programming by carefully analyzing the probability that a given item has been picked before
- If $\Psi = \mathrm{PL}_\nu$, no polynomial-time algorithm known so far...



Some examples

Example of values of $v(k,t)$ for $\Psi = IC$ (Borda scoring function):

$k \backslash t$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	7	6.86	6.67	6.4	6	5.33	4	-
2	13	12.57	12	11.2	10	8	-	-
3	18	17.14	16	14.4	12	-	-	-
4	22	20.57	18.67	16	-	-	-	-
5	25	22.86	20	-	-	-	-	-
6	27	24	-	-	-	-	-	-
7	28	-	-	-	-	-	-	-

Note: the approach is similar to that of [Kalinowski et al., 2013]



Kalinowski, T., Narodytska, N., and Walsh, T. (2013).
A social welfare optimal sequential allocation procedure.
In *Proceedings of IJCAI 2013.*



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Let us put things together and formalize...



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Let $\widehat{SW}^F_\Psi(i, \ell)$ =the best expected utility we can obtain for agents $i, \ldots, n$ if ℓ items have already been picked.



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$$\widehat{SW}^F_\Psi(i,\ell) = \max_{k\in\{0,\ldots,m-\ell\}} F\left(v(k,\ell), \widehat{SW}^F_\Psi(i+1,\ell+k)\right)$$

(note: $F \in \{+, \min, \times\}$)



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(note: $F \in \{+, \min, \times\}$)

$$\widehat{SW}^F_\Psi(n,\ell) = v(m-\ell,\ell)$$



Computation time

Main result

If the key property is met, for any scoring function g, and any $F \in \{\min, +, \times\}$, we can find in time $O(nm^2K(n,m,g))$ the vector $(k_1, \ldots, k_n)$ that maximizes $SW_\Psi^F(\overrightarrow{k})$, if $K(n,m,g)$ is the time needed to compute $EU_\Psi^{\overrightarrow{k}}(a)$.

$\Rightarrow$ polynomial time for *IC* and *FC*.



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$\Rightarrow$ polynomial time for *IC* and *FC*.

And for PL_ν?
No known polynomial-time algorithm, but $EU_\Psi^{\overrightarrow{k}}(a)$ can be efficiently approximated by sampling, with approximation guarantees (Hoeffding's inequality).



Beyond the key property

What if the key property is not met (*e.g* $\mathsf{MII}_{\mu,\phi}$)?



Beyond the key property

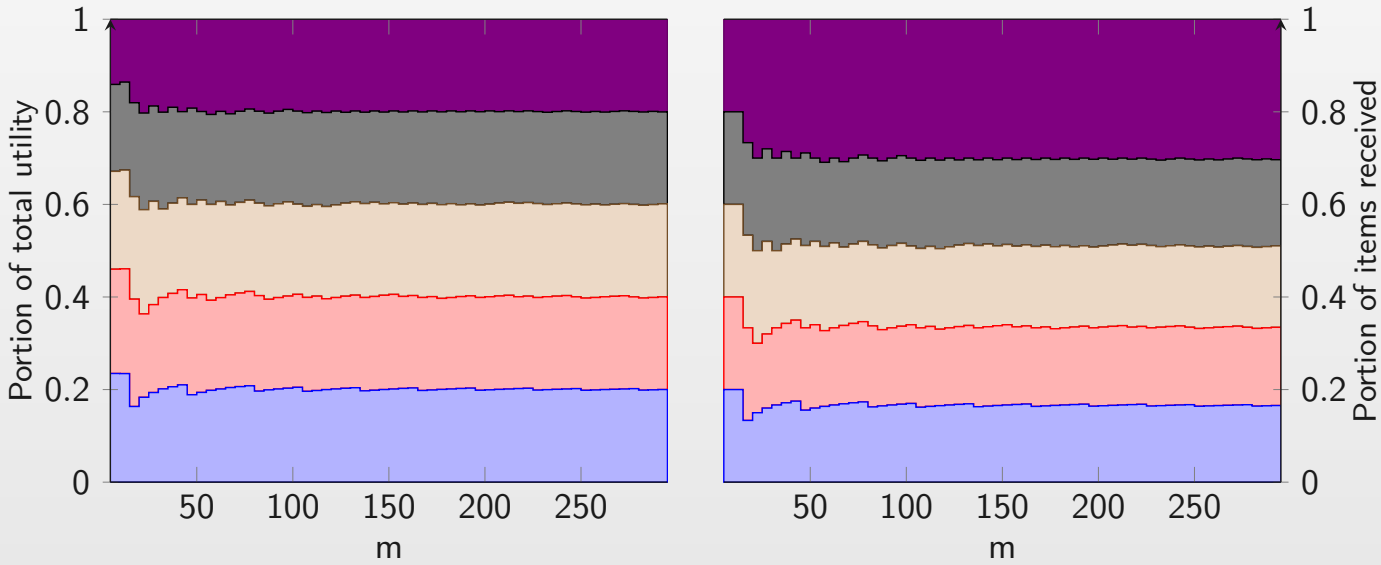
What if the key property is not met (*e.g* $\mathrm{MII}_{\mu,\phi}$)?

- Good news: for **egalitarian SW**, a greedy-like algorithm can compute the best ESW in time $O(nmK(n,m,g))$
- Open problem: for **Nash and utilitarian SW**, we do not know any efficient algorithm so far
- Bad news: for **utilitarian SW**, sampling the profile and computing the best $\overrightarrow{k}$ over this profile is **NP**-complete



Some results (egalitarian SW)

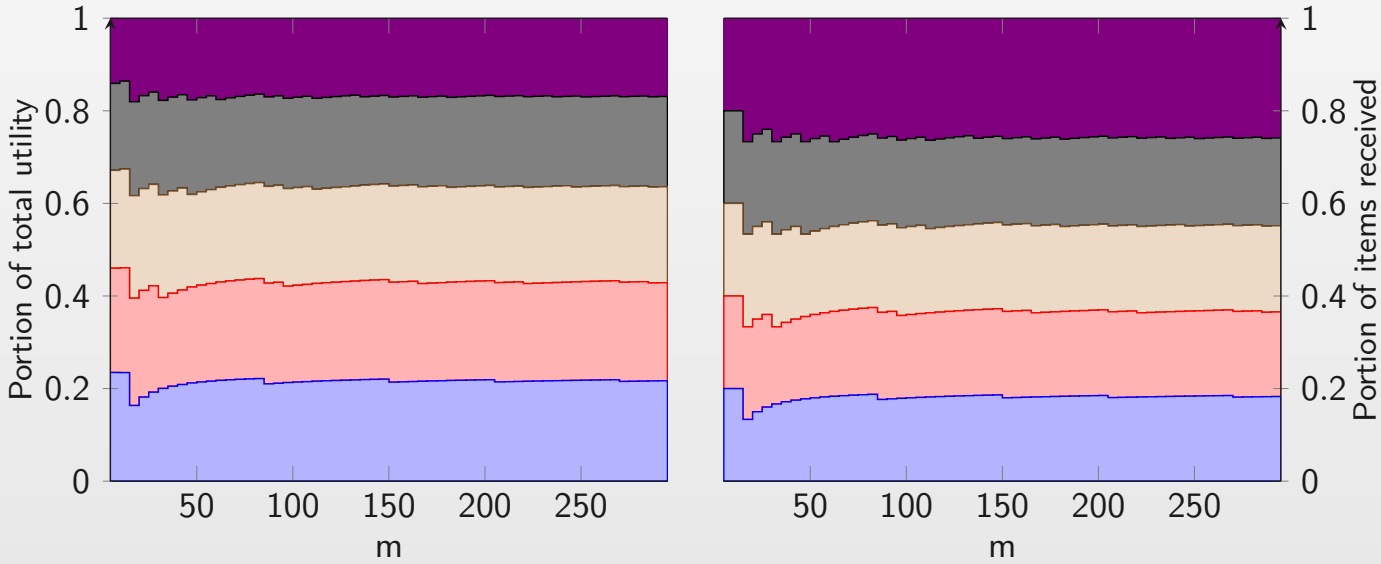
5 agents, varying # of goods, Borda scoring vector, $\Psi = IC$, $F = \min$





Some results (Nash SW)

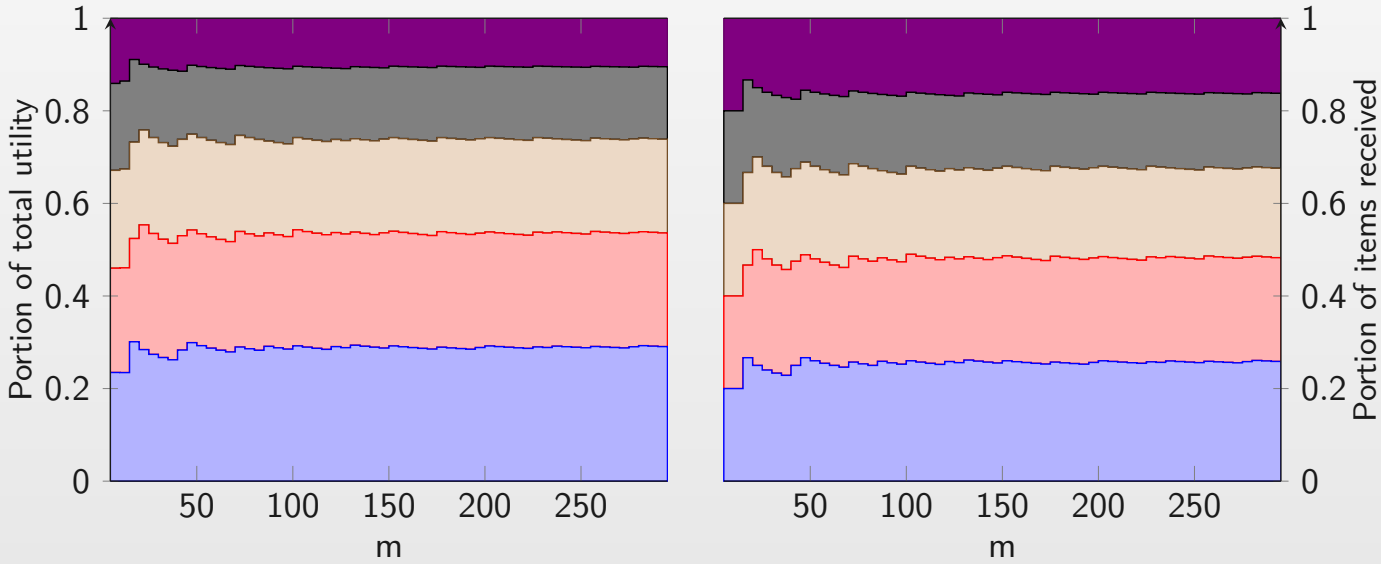
5 agents, varying # of goods, Borda scoring vector, $\Psi = IC$, $F = \times$





Some results (utilitarian SW)

5 agents, varying # of goods, Borda scoring vector, $\Psi = IC$, $F = +$



Do you see anything strange?

Computational barriers to manipulation

Can we rely on computational complexity to prevent manipulation?



Complexity as a barrier to manipulation

Let us go back to manipulable picking sequences (general ones)...



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First problem:

A: *"Can I get $\mathcal{S}$ for sure?"*



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Getting a subset for sure

We can answer to that constructively in polynomial time!



Complexity as a barrier to manipulation

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Getting a subset for sure

We can answer to that constructively in polynomial time!

Idea:

- two agents: pick the objects in $\mathcal{S}$ in the same order of $\succ_B$
- more agents:
  - transform agents 2 to $m-1$ into a single (fake) agent
  - apply the algorithm for 2 agents



General manipulation problem

A: *"What is the best subset I can get?"*



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Idea: Greedily build the optimal achievable subset:

- Find the best object i such that $\{i\}$ is achievable;
- Find the best object j such that $\{i, j\}$ is achievable;
- ...



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Manipulation with additive preferences, two agents

If the manipulator has additive preferences, the optimal manipulation can be computed in polynomial time.



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Manipulation with additive preferences, two agents

If the manipulator has additive preferences, the optimal manipulation can be computed in polynomial time.

Only works **for two agents**!



General manipulation problem

Result [Aziz et al., 2017]

If the manipulator has additive preferences, the optimal manipulation problem is **NP**-complete.

(reduction from [3-SAT])

- Is there a manipulation that yields a better utility than the truthful report? $\rightsquigarrow$ **NP**-complete
- Not true anymore for binary utilities and (ordinal) responsive set extension.

Aziz, H., Bouveret, S., Lang, J., and Mackenzie, S. (2017).
Complexity of manipulating sequential allocation.
In *Proceedings of the 31st AAAI conference on Artificial Intelligence (AAAI'17).*



Coalitional Manipulation

Example

3 agents, 6 objects:

- A: $1 \succ 2 \succ 5 \succ 4 \succ 3 \succ 6$
- B: $1 \succ 3 \succ 5 \succ 2 \succ 4 \succ 6$
- C: $2 \succ 3 \succ 4 \succ 1 \succ 5 \succ 6$

Sequence $\pi = ABCABC \rightarrow \{15|34|26\}$.



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Sequence $\pi = ABCABC \rightarrow \{15|34|26\}$.

- If A and B manipulate alone, they cannot do better
- If they cooperate, they can get $\{12|35|46\}$, which is strictly better.



Coalitional Manipulation: Results

Three kinds of manipulation considered here:

- No post-allocation trade allowed between the manipulators
- Post-allocation exchange of goods allowed between the manipulators
- Post-allocation exchange of goods + side-payments allowed



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Results:

- No post-allocation trade allowed between the manipulators → **NP**-complete [PARTITION]
- Post-allocation exchange of goods allowed between the manipulators → **NP**-complete [PARTITION]
- Post-allocation exchange of goods + side-payments allowed → polynomial (comes down to manipulation by a single agent)



Everyone manipulates...

One manipulator



Everyone manipulates...

One manipulator $\rightarrow$ several manipulators (coalitional manipulation)



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Game Theory (Subgame Perfect Nash Equilibrium)



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One manipulator $\rightarrow$ several manipulators (coalitional manipulation)
$\rightarrow$ everyone (rational, self-interested) manipulates?

Game Theory (Subgame Perfect Nash Equilibrium)

- Two agents and additive utilities, precise characterization of the result of every SPNE $((rev(\succ_2), rev(\succ_1), rev(\pi)))$
  [Kalinowski et al., 2013, Kohler and Chandrasekaran, 1971].
- Unbounded number of agents: **PSPACE**-hard [Kalinowski et al., 2013].

Kalinowski, T., Narodytska, N., Walsh, T., and Xia, L. (2013).
Strategic behavior when allocating indivisible goods sequentially.
In *Proceedings of AAAI'13*.

Kohler, D. A. and Chandrasekaran, R. (1971).
A class of sequential games.
Operations Research, 19(2):270–277.

Conclusion

A take-away message?



Conclusion

- A simple and intuitive sequential allocation procedure (actually already known in sparse litterature)
- Finding the best policy?
  - Full Correlation case well understood
  - Impartial Culture: partial (hardness) results
- Strategical issues:
  - Non-interleaved sequences: strategyproof, and can be fair!
  - Complexity as a barrier to (individual and coalitional) manipulation



A note about efficiency

As soon as we constrain serial dictatorship, we lose Pareto-efficiency.



A note about efficiency

As soon as we constrain serial dictatorship, we lose Pareto-efficiency.
But:

- Any Pareto-efficient allocation is sequenceable
- The converse is true for
  - **quantity-monotonic** preferences [Pápai, 2000]
  - **lexicographic** preferences [Hosseini and Larson, 2019]
- Sequenceable allocations correspond to a weak form of Pareto-optimal ones
- $\rightarrow$ Sequenceability $\approx$ weak (local) form of efficiency

Hosseini, H. and Larson, K. (2019).
Multiple assignment problems under lexicographic preferences.
In *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS '19*, pages 837–845.

Pápai, S. (2000).
original papers : Strategyproof multiple assignment using quotas.
Review of Economic Design, 5(1):91–105.



Very nice properties...

A simple protocol, but with nice features:

- "locally" efficient
- efficient with respect to cycle deals
- guarantees envy-freeness up to one good
- gives good approximation of social welfare
- also gives good approximation of other fairness properties (*e.g.* max-min share)