Characterizing Conflicts in Fair Division of Indivisible Goods Using a Scale of Criteria

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1 2 3

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agent 2

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$u_2(\{2,3\}) = 1 + 6 = 7$

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$\overrightarrow{\pi} = \langle\{1\}, \{2, 3\}\rangle$

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In this work, we consider five **fairness** criteria.

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$\overrightarrow{\pi}' = \langle\{1, 2\}, \{3\}\rangle$ is envy-free.

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Known facts:

- An envy-free allocation may not exist.
- Deciding whether an allocation is envy-free is easy (quadratic time).
- Deciding whether an instance (agents, objects, preferences) has an envy-free allocation is hard – NP-complete [Lipton et al., 2004].

Lipton, R., Markakis, E., Mossel, E., and Saberi, A. (2004).
On approximately fair allocations of divisible goods.
In *Proceedings of EC'04*.

Proportional fair share

Proportional fair share (PFS):

- Initially defined by [Steinhaus, 1948] for continuous fair division (*cake-cutting*)
- **Idea:** each agent is "entitled" to at least the nth of the entire resource

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The **proportional fair share** of an agent i is equal to:

$$u_i^{\text{PFS}} \stackrel{def}{=} \frac{u_i(\mathcal{O})}{n} = \sum_{o \in \mathcal{O}} \frac{w_i(o)}{n}$$

An allocation $\vec{\pi}$ satisfies **(proportional) fair share** if every agent gets at least her fair share.

Proportional fair share: facts

Easy or known facts:

- Deciding whether an allocation satisfies proportional fair share (PFS) is easy (linear time).
- For a given instance, there may be no allocation satisfying PFS
  → *e.g.* 2 agents, 1 object
- This is not true for cake-cutting (divisible resource)
  → Dubins-Spanier

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New (?) facts:

- Deciding whether an instance has an allocation satisfying PFS is hard even for 2 agents – **NP**-complete [PARTITION].
- $\overrightarrow{\pi}$ is envy-free $\Rightarrow$ $\overrightarrow{\pi}$ satisfies PFS[1].

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Max-min fair share (MFS):

- Introduced recently [Budish, 2011]; not so much studied so far.
- **Idea:** in the **cake-cutting** case, PFS = the best share an agent can hopefully get for sure in a *"I cut, you choose (I choose last)"* game.
- Same game for indivisible goods → MFS.

Budish, E. (2011).

The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes.
Journal of Political Economy, 119(6).

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$$u_i^{\text{MFS}} \overset{def}{=} \max_{\overrightarrow{\pi}} \min_{j \in \mathcal{A}} u_i(\pi_j)$$

An allocation $\overrightarrow{\pi}$ satisfies **max-min fair share** (MFS) if every agent gets at least her max-min fair share.

Max-min fair share: examples

Example: 3 objects $\{1, 2, 3\}$, 2 agents $\{1, 2\}$.

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agent 1	5	4	2	$\rightarrow u_1^{\mathrm{MFS}} = 5$ (with cut $\langle\{1\}, \{2, 3\}\rangle$)
agent 2	4	1	6	$\rightarrow u_2^{\mathrm{MFS}} = 5$ (with cut $\langle\{1, 2\}, \{3\}\rangle$)

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MFS evaluation:

$\overrightarrow{\pi} = \langle \{1\}, \{2, 3\} \rangle \rightarrow u_1(\pi_1) = 5 \geq 5;\ u_2(\pi_2) = 7 \geq 5 \Rightarrow$ MFS satisfied

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Example: 2 agents, 1 object.

$u_1^{\text{MFS}} = u_2^{\text{MFS}} = 0 \rightarrow$ every allocation satisfies MFS!

Not very satisfactory, but can we do much better?

Max-min fair share: properties

Facts:

- Computing u_i^{MFS} for a given agent is hard $\rightarrow$ **NP**-complete [PARTITION]
- Hence, deciding whether an allocation satisfies MFS is also hard.
- $\overrightarrow{\pi}$ satisfies PFS $\Rightarrow$ $\overrightarrow{\pi}$ satisfies MFS.

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- No counterexample found on **thousands** of **random** instances.

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Conjecture

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FALSE!

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- No counterexample found on **thousands** of **random** instances.

The conjecture has been proved **false** by Procaccia and Wang using a **very tricky** counterexample.

Procaccia, A. D. and Wang, J. (2014).
Fair enough: Guaranteeing approximate maximin shares.
In *Proc. 14th ACM Conference on Economics and Computation (EC'14)*.
forthcoming.

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- mFS = the worst share an agent can get in a *"Someone cuts, I choose first"* game.
- In the **cake-cutting** case, same as PFS.

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- Hence, deciding whether an allocation satisfies mFS is also hard.
- $\overrightarrow{\pi}$ satisfies mFS $\Rightarrow$ $\overrightarrow{\pi}$ satisfies PFS.
- $\overrightarrow{\pi}$ is envy-free $\Rightarrow$ $\overrightarrow{\pi}$ satisfies mFS.

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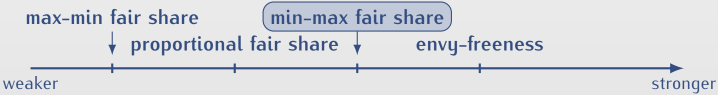

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Disjoint shares!

$\Rightarrow$ Allocation $\langle\{1, 4\}, \{2, 3\}\rangle$, with prices $\langle 0.8, 0.2, 0.8, 0.2\rangle$ forms a CEEI.

$\Rightarrow$ Allocation $\langle\{1, 4\}, \{2, 3\}\rangle$ satisfies CEEI.

Competitive Equilibrium from Equal Incomes

- Classical notion in economics [Moulin, 1995]
- Not so much studied in computer science (except [Othman et al., 2010])

Moulin, H. (1995).
Cooperative Microeconomics, A Game-Theoretic Introduction.
Prentice Hall.

Othman, A., Sandholm, T., and Budish, E. (2010).
Finding approximate competitive equilibria: efficient and fair course allocation.
In *Proceedings of AAMAS'10.*

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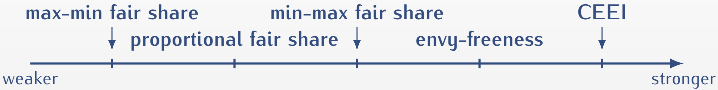

Interpretation

max-min fair share ↓ proportional fair share — min-max fair share ↓ envy-freeness — CEEI ↓

weaker ⟶ stronger

1 For all allocation $\overrightarrow{\pi}$:

$$(\overrightarrow{\pi} \models \text{CEEI}) \Rightarrow (\overrightarrow{\pi} \models \text{EF}) \Rightarrow (\overrightarrow{\pi} \models \text{mFS}) \Rightarrow (\overrightarrow{\pi} \models \text{PFS}) \Rightarrow (\overrightarrow{\pi} \models \text{MFS})$$

$\rightarrow$ the highest property $\overrightarrow{\pi}$ satisfies, the most satisfactory it is.

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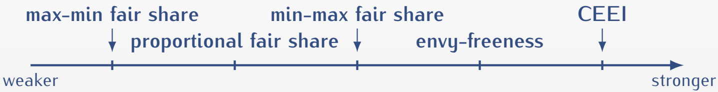

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$$\mathcal{I}_{|\text{CEEI}} \subset \mathcal{I}_{|\text{EF}} \subset \mathcal{I}_{|\text{mFS}} \subset \mathcal{I}_{|\text{PFS}} \subset \mathcal{I}_{|\text{MFS}} \subset \mathcal{I}$$

$\rightarrow$ the lowest subset, the less “conflict-prone”.

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Two extreme examples:

- 2 agents, 1 object $\rightarrow$ only in $\mathcal{I}_{|\text{MFS}}$
- 2 agents, 2 objects, with

	1	2
agent 1	1000	0
agent 2	0	1000

$\rightarrow$ in $\mathcal{I}_{|\text{CEEI}}$ (with *e.g.* $\overrightarrow{p} = \langle 1, 1 \rangle$).

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A scale of properties (for numerical additive preferences)...

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Almost always possible to satisfy it

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A possible approach to fairness in multiagent resource allocation problems:

1. Determine the highest satisfiable criterion.
2. Find an allocation that satisfies this criterion.
3. Explain to the upset agents that we cannot do much better.

What else

Some other results (see the paper)...

- All the inclusions are strict
- Link with egalitarianism
- Experimental results
- A glimpse beyond additive preferences

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- A glimpse beyond additive preferences

Some future directions...

- Some missing complexity results.
- Develop efficient **algorithms**
- More **experiments**
- Extend to **more expressive preference languages** (including ordinal ones...).