Picking Sequences for Resource Allocation
A (not so) Short Introduction

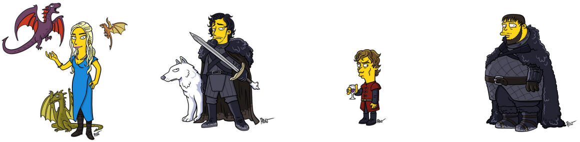
Sylvain Bouveret
LIG – Univ. Grenoble-Alpes

Summer School on the Engineering Aspects of Economic Design

Budapest, June the 10th, 2019

Fair division of indivisible goods...

Fair division of indivisible goods...



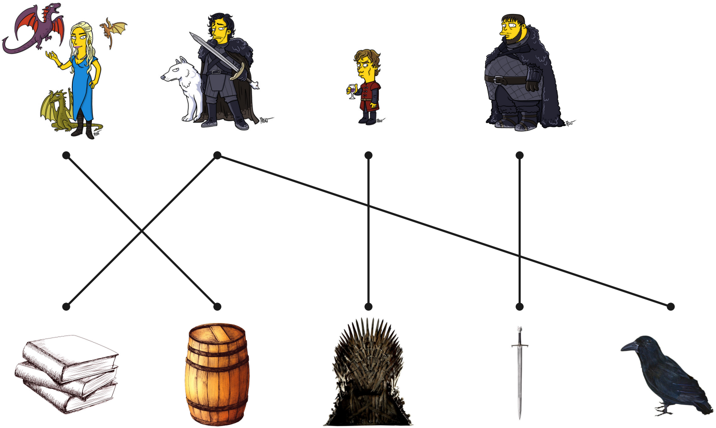
Fair division of indivisible goods...








Fair division of indivisible goods...



How to solve this problem

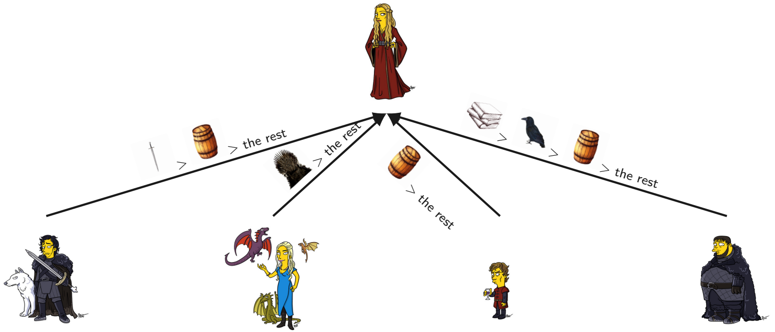
1. Ask the agents to give their preferences and use a (centralized) collective decision making procedure.

How to solve this problem

1 Ask the agents to give their preferences and use a (centralized) collective decision making procedure.

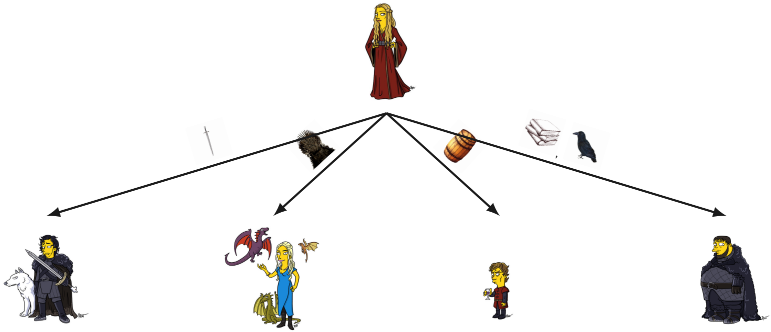
How to solve this problem

1 Ask the agents to give their preferences and use a (centralized) collective decision making procedure.



How to solve this problem

1 Ask the agents to give their preferences and use a (centralized) collective decision making procedure.



How to solve this problem

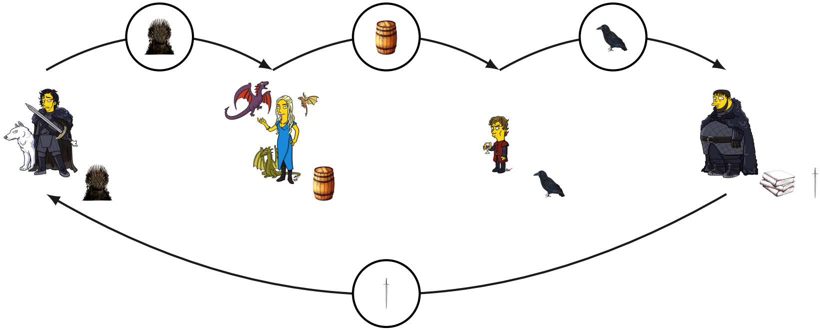
1. Ask the agents to give their preferences and use a (centralized) collective decision making procedure.
2. Start from a random allocation and ask the agents to negotiate.

How to solve this problem

1. Ask the agents to give their preferences and use a (centralized) collective decision making procedure.
2. Start from a random allocation and ask the agents to negotiate.

How to solve this problem

1. Ask the agents to give their preferences and use a (centralized) collective decision making procedure.
2. Start from a random allocation and ask the agents to negotiate.

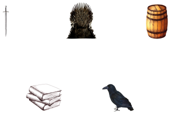


How to solve this problem

1. Ask the agents to give their preferences and use a (centralized) collective decision making procedure.
2. Start from a random allocation and ask the agents to negotiate.
3. Use an interactive protocol like **picking sequences**.

How to solve this problem

1. Ask the agents to give their preferences and use a (centralized) collective decision making procedure.
2. Start from a random allocation and ask the agents to negotiate.
3. Use an interactive protocol like **picking sequences**.



How to solve this problem

1. Ask the agents to give their preferences and use a (centralized) collective decision making procedure.
2. Start from a random allocation and ask the agents to negotiate.
3. Use an interactive protocol like **picking sequences**.

How to solve this problem

1 | Ask the agents to give their preferences and use a (centralized) collective decision making procedure.

2 | Start from a random allocation and ask the agents to negotiate.

3 | Use an interactive protocol like **picking sequences**.

How to solve this problem

1. Ask the agents to give their preferences and use a (centralized) collective decision making procedure.
2. Start from a random allocation and ask the agents to negotiate.
3. Use an interactive protocol like **picking sequences**.

How to solve this problem

1. Ask the agents to give their preferences and use a (centralized) collective decision making procedure.
2. Start from a random allocation and ask the agents to negotiate.
3. Use an interactive protocol like **picking sequences**.

How to solve this problem

1 | Ask the agents to give their preferences and use a (centralized) collective decision making procedure.

2 | Start from a random allocation and ask the agents to negotiate.

3 | Use an interactive protocol like **picking sequences**.

How to solve this problem

1. Ask the agents to give their preferences and use a (centralized) collective decision making procedure.
2. Start from a random allocation and ask the agents to negotiate.
3. Use an interactive protocol like **picking sequences**.

In this presentation, we will focus on **picking sequences** (but talk a little bit about the rest)

More formally. . .

Ask the individuals to pick in turn their most preferred object among the remaining ones, according to some **predefined sequence**.

Example

3 individuals A, B, C, 6 items, sequence $ABCCBA \rightarrow A$ chooses first (and takes her preferred item), then B, then C, then C again. . .

Sequences

- Natural and simple protocol
- Used in practice
- Preference elicitation-free

- Board games
- Draft mechanisms (sport)
- Course allocation (Harvard Business School)
- ...

What I will present today...

1. **Optimal sequence:** We “feel” that *ABCCBA* is **fairer** than *AABBCC*...
   $\rightarrow$ *What is the **fairest** sequence ?*
2. **Manipulation:** What if the agents act strategically?
3. **Efficiency:** Picking sequences as an optimization procedure.

The optimal sequence problem

What is the fairest sequence?

Agents, objects, preferences. . .

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A central authority (CA) must find a **policy** (a sequence of agents) $\pi : \{1, \ldots, p\} \rightarrow \{A, B, \ldots, x\}$
- The central authority assumes that each agent i has a (private) **ranking** $\succ_i$ over $\mathcal{O}$ (ex: $6 \succ 1 \succ 4 \succ 5 \succ 2 \succ 3$)

Example

Example

5 objects, 3 agents, $\pi = ABCCB\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

Example

Example

5 objects, 3 agents, $\pi = ABCCB\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

k	0
$s(A)_k^\pi$	$\emptyset$
$s(B)_k^\pi$	$\emptyset$
$s(C)_k^\pi$	$\emptyset$
$\mathcal{O}_k^\pi$	$\emptyset$

Example

Example

5 objects, 3 agents, $\pi = \mathbf{A}BCCB\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

k	0	1
$s(A)_k^\pi$	$\emptyset$	1
$s(B)_k^\pi$	$\emptyset$	$\emptyset$
$s(C)_k^\pi$	$\emptyset$	$\emptyset$
$\mathcal{O}_k^\pi$	$\emptyset$	1

Example

Example

5 objects, 3 agents, $\pi = A\mathbf{B}CCB\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

k	0	1	2
$s(A)_k^\pi$	$\emptyset$	1	1
$s(B)_k^\pi$	$\emptyset$	$\emptyset$	4
$s(C)_k^\pi$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathcal{O}_k^\pi$	$\emptyset$	1	14

Example

Example

5 objects, 3 agents, $\pi = AB\mathbf{C}CB\dots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

k	0	1	2	3
$s(A)^\pi_k$	$\emptyset$	1	1	1
$s(B)^\pi_k$	$\emptyset$	$\emptyset$	4	4
$s(C)^\pi_k$	$\emptyset$	$\emptyset$	$\emptyset$	3
$\mathcal{O}^\pi_k$	$\emptyset$	1	14	143

Example

Example

5 objects, 3 agents, $\pi = ABC\mathbf{C}B\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

k	0	1	2	3	4
$s(A)_k^\pi$	$\emptyset$	1	1	1	1
$s(B)_k^\pi$	$\emptyset$	$\emptyset$	4	4	4
$s(C)_k^\pi$	$\emptyset$	$\emptyset$	$\emptyset$	3	35
$\mathcal{O}_k^\pi$	$\emptyset$	1	14	143	1435

Example

Example

5 objects, 3 agents, $\pi = ABCC\mathbf{B}\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

k	0	1	2	3	4	5
$s(A)^{\pi}_{k}$	$\emptyset$	1	1	1	1	1
$s(C)^{\pi}_{k}$	$\emptyset$	$\emptyset$	4	4	4	42
$s(C)^{\pi}_{k}$	$\emptyset$	$\emptyset$	$\emptyset$	3	35	35
$\mathcal{O}^{\pi}_{k}$	$\emptyset$	1	14	143	1435	14352

Scoring functions

We only have rankings over objects. . .

$\rightarrow$ *How to compare two allocations ?*

Scoring functions

We only have rankings over objects...

$\rightarrow$ *How to compare two allocations ?*

Two natural assumptions:

1. **Scoring:** We have a common **scoring function** $g : \{1, \ldots, p\} \mapsto \mathbb{N}$ mapping each rank to a utility.
2. **Additivity:** These utilities are **additive**.

Scoring functions

We only have rankings over objects...

→ *How to compare two allocations ?*

Two natural assumptions:

1. **Scoring:** We have a common **scoring function** $g : \{1, \ldots, p\} \mapsto \mathbb{N}$ mapping each rank to a utility.
2. **Additivity:** These utilities are **additive**.

3 natural scoring functions:

$\succ_i$	6	1	4	5	2	3

Scoring functions

We only have rankings over objects...

→ *How to compare two allocations ?*

Two natural assumptions:

1. **Scoring:** We have a common **scoring function** $g : \{1, \dots, p\} \mapsto \mathbb{N}$ mapping each rank to a utility.
2. **Additivity:** These utilities are **additive**.

3 natural scoring functions:

$\succ_i$	6	1	4	5	2	3
Borda	6	5	4	3	2	1

Scoring functions

We only have rankings over objects. . .

→ *How to compare two allocations ?*

Two natural assumptions:

1. **Scoring:** We have a common **scoring function** $g : \{1, \ldots, p\} \mapsto \mathbb{N}$ mapping each rank to a utility.
2. **Additivity:** These utilities are **additive**.

3 natural scoring functions:

$\succ_i$	6	1	4	5	2	3
Borda	6	5	4	3	2	1
lexicographic	32	16	8	4	2	1

Scoring functions

We only have rankings over objects...

$\rightarrow$ *How to compare two allocations ?*

Two natural assumptions:

1. **Scoring:** We have a common **scoring function** $g : \{1, \dots, p\} \mapsto \mathbb{N}$ mapping each rank to a utility.
2. **Additivity:** These utilities are **additive**.

3 natural scoring functions:

$\succ_i$	6	1	4	5	2	3
Borda	6	5	4	3	2	1
lexicographic	32	16	8	4	2	1
Quasi-Indifference	$1+5\varepsilon$	$1+4\varepsilon$	$1+3\varepsilon$	$1+2\varepsilon$	$1+\varepsilon$	1

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB\ldots$

- $A : \mathbf{1} \succ 2 \succ 3 \succ 4 \succ 5$
- $B : \mathbf{4} \succ \mathbf{2} \succ 5 \succ 1 \succ 3$
- $C : 1 \succ \mathbf{3} \succ \mathbf{5} \succ 4 \succ 2$

With π, agent A gets 1, agent B gets 24, agent C gets 35

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB\ldots$

- $A : \mathbf{1} \succ 2 \succ 3 \succ 4 \succ 5$
- $B : \mathbf{4} \succ \mathbf{2} \succ 5 \succ 1 \succ 3$
- $C : 1 \succ \mathbf{3} \succ \mathbf{5} \succ 4 \succ 2$

With π, agent A gets 1, agent B gets 24, agent C gets 35

- **Borda:** $u_A(\pi) = 5$; $u_B(\pi) = 5 + 4 = 9$; $u_C(\pi) = 4 + 3 = 7$.
- **lexicographic:** $u_A(\pi) = 16$; $u_B(\pi) = 24$; $u_C(\pi) = 12$.
- **QI:** $u_A(\pi) = 1 + 4\varepsilon$; $u_B(\pi) = 2 + 7\varepsilon$; $u_C(\pi) = 2 + 5\varepsilon$.

Social welfare

We use a **collective utility function** to aggregate the individual utilities.

Two well-known functions:

- **utilitarian:** $F(u_A, \ldots, u_x) = \sum_{i \in \mathcal{N}} u_i$;
- **egalitarian:** $F(u_A, \ldots, u_x) = \min_{i \in \mathcal{N}} u_i$.

Uncertainty

The procedure is elicitation-free...

$\rightarrow$ *Which information can we use to find the best sequence ?*

Uncertainty

The procedure is elicitation-free...

→ *Which information can we use to find the best sequence ?*

The CA has a **probabilistic model** of the preferences:

- **Full independence** : each profile $R = \langle \succ_A, \ldots, \succ_x \rangle$ is equally probable
- **Full correlation** : all the agents have the same ranking $(R = \langle \succ, \ldots, \succ \rangle)$

Uncertainty

The procedure is elicitation-free...

→ *Which information can we use to find the best sequence ?*

The CA has a **probabilistic model** of the preferences:

- **Full independence** : each profile $R = \langle \succ_A, \ldots, \succ_x \rangle$ is equally probable
- **Full correlation** : all the agents have the same ranking $(R = \langle \succ, \ldots, \succ \rangle)$

Expected individual and collective utilities:

$$\overline{u(i,\pi)} = \sum_{R \in Prof(\mathcal{N},\mathcal{O})} Pr(R) \times u_i(\pi, R).$$

$$\overline{sw_F(\pi)} = F(\overline{u(1,\pi)}, \ldots, \overline{u(n,\pi)}).$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.
What is agent 3's expected utility with this sequence ?

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.

What is agent 3's expected utility with this sequence ?

3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3, \pi)} =$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.

What is agent 3's expected utility with this sequence ?

3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3,\pi)} = \frac{1}{\binom{5}{2}} \times (3+2)$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.

What is agent 3's expected utility with this sequence ?

3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3,\pi)} = 0.5 + \frac{1}{\binom{5}{2}} \times (4+2)$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.

What is agent 3's expected utility with this sequence ?

3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3,\pi)} = 0.5 + 0.6 + \frac{1}{\binom{5}{2}} \times (5+2)$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.
What is agent 3's expected utility with this sequence ?
3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3,\pi)} = 0.5 + 0.6 + 0.6 + \frac{2}{\binom{5}{2}} \times (4 + 3)$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.

What is agent 3's expected utility with this sequence ?

3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3,\pi)} = 0.5 + 0.6 + 0.6 + 1.4 + \frac{2}{\binom{5}{2}} \times (5+3)$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.

What is agent 3's expected utility with this sequence ?

3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3,\pi)} = 0.5 + 0.6 + 0.6 + 1.4 + 1.6 + \frac{\binom{3}{2}}{\binom{5}{2}} \times (5+4)$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.
What is agent 3's expected utility with this sequence ?
3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3,\pi)} = 0.5 + 0.6 + 0.6 + 1.4 + 1.6 + 2.7 = \mathbf{7.5}$$

Summary

- **Instance:**
  - a number of agents n
  - a number of objects p
  - a scoring function g
  - a correlation profile $Corr \in \{FC, FI\}$
  - a collective utility function F
- **Question:**
  - *What is the policy* π *maximizing* $\overline{sw_F(\pi)}$, *under correlation profile Corr ?*

Some general results

1. **Full correlation**

Some general results

1. Full correlation

Utilitarian CUF (sum)

All policies have the same expected value!

Some general results

1. Full correlation

Utilitarian CUF (sum)

All policies have the same expected value!

Egalitarian CUF (min)

Sequential allocation is **NP**-complete.

(Reduction from [PARTITION])

Some general results

1. Full correlation

Utilitarian CUF (sum)

All policies have the same expected value!

Egalitarian CUF (min)

Sequential allocation is **NP**-complete.

(Reduction from [PARTITION])

What about...

- ...lexicographic scoring ?
- ...quasi-indifference scoring ?
- ...Borda scoring ?

Lexicographic scoring

$\succ_i$	6	$\succ$	1	$\succ$	4	$\succ$	5	$\succ$	2	$\succ$	3
lexicographic	32	$>>$	16	$>>$	8	$>>$	4	$>>$	2	$>>$	1

Egalitarian CUF (min)

Optimal policies: $\sigma(A)\sigma(B)\ldots\sigma(x)\sigma(x)^{p-n}$ (where σ is a permutation of $\{A, B, \ldots, x\}$)

Lexicographic scoring

$\succ_i$	6	$\succ$	1	$\succ$	4	$\succ$	5	$\succ$	2	$\succ$	3
lexicographic	32	$>>$	16	$>>$	8	$>>$	4	$>>$	2	$>>$	1

Egalitarian CUF (min)

Optimal policies: $\sigma(A)\sigma(B)\ldots\sigma(x)\sigma(x)^{p-n}$ (where σ is a permutation of $\{A, B, \ldots, x\}$)

Example: $\pi = ABCCCC$

- $\overline{u(1, \pi)} = 32$
- $\overline{u(2, \pi)} = 16$
- $\overline{u(3, \pi)} = 8 + 4 + 2 + 1 = 15$

Borda scoring

$\succ_i$	6	1	4	5	2	3
Borda	6	5	4	3	2	1

Borda scoring

$\succ_i$	6	1	4	5	2	3
Borda	6	5	4	3	2	1

This is **polynomial** in p (dynamic programming algorithm).

QI scoring

$\succ_i$	6	1	4	5	2	3
Quasi-Indifference	$1+5\varepsilon$	$1+4\varepsilon$	$1+3\varepsilon$	$1+2\varepsilon$	$1+\varepsilon$	1

Egalitarian CUF (min)

Comes down to solving the Borda case!

QI scoring

$\succ_i$	6	1	4	5	2	3
Quasi-Indifference	$1+5\varepsilon$	$1+4\varepsilon$	$1+3\varepsilon$	$1+2\varepsilon$	$1+\varepsilon$	1

Egalitarian CUF (min)

Comes down to solving the Borda case!

Intuition:

- let $m = \lfloor \frac{p}{n} \rfloor$ and $q = p - nm$
- Optimal policies: $\pi = \underbrace{AABB}_{n-q \text{ agents}} \overbrace{CCCDDD}^{q \text{ agents}}$ and $\pi' = \underbrace{ABBA}_{n-q \text{ agents}} \overbrace{CCCDDD}^{q \text{ agents}}$
- The q last agents are OK $\rightarrow u \geq m+1$
- The $n-q$ first agents: $u = m + x \cdot \varepsilon$ ($x \rightarrow$ Borda)

A complex problem. . .

2. **Full independence**

A complex problem. . .

2. Full independence

Conjecture (2011)

Computing the expected utility of a sequence is **NP**-complete.
Computing the **optimal** sequence probably harder.

Results

Computing the expected utility of a sequence is **NP**-complete.

Results

Computing the expected utility of a sequence is ~~NP-complete~~ **polynomial** [Kalinowski et al., 2013].

Kalinowski, T., Narodytska, N., and Walsh, T. (2013).
A social welfare optimal sequential allocation procedure.
In *Proceedings of IJCAI 2013*.

Results

Computing the expected utility of a sequence is ~~**NP**-complete~~ **polynomial** [Kalinowski et al., 2013].

Computing the **optimal** sequence probably harder.

Kalinowski, T., Narodytska, N., and Walsh, T. (2013).
A social welfare optimal sequential allocation procedure.
In *Proceedings of IJCAI 2013*.

Results

Computing the expected utility of a sequence is ~~**NP**-complete~~ **polynomial** [Kalinowski et al., 2013].

Computing the **optimal** sequence probably harder.

- the **alternating policy** (*ABABABAB*...) is optimal for Borda, utilitarian social welfare
- complexity unknown for other social welfare and scoring functions (**NP**-hardness conjectured)

Kalinowski, T., Narodytska, N., and Walsh, T. (2013).
A social welfare optimal sequential allocation procedure.
In *Proceedings of IJCAI 2013*.

Some examples

Assumptions: Full independence, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4		
5		
6		
8		
10		

Some examples

Assumptions: Full independence, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4	ABBA	
5		
6		
8		
10		

Some examples

Assumptions: Full independence, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4	ABBA	ABCC
5		
6		
8		
10		

Some examples

Assumptions: Full independence, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4	ABBA	ABCC
5	AABBB	ABCCB
6		
8		
10		

Some examples

Assumptions: Full independence, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4	ABBA	ABCC
5	AABBB	ABCCB
6	ABABBA	ABCCBA
8	ABBABAAB	AACCBBCB
10	ABBAABABBA	ABCABBCACC

Some examples

Assumptions: Full independence, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4	ABBA	ABCC
5	AABBB	ABCCB
6	ABABBA	ABCCBA
8	ABBABAAB	AACCBBCB
10	ABBAABABBA	ABCABBCACC

Other examples on
`http://recherche.noiraudes.net/en/sequences.php`

Strategical issues (manipulation)

Is the protocol strategy-proof?

Manipulation?

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A central authority (CA) has chosen a **policy** π and will execute it
- The agents have their own private preferences $\rightarrow$ **picking strategy**.

Manipulation?

Example

2 agents, 4 objects:

- A: $1 \succ 2 \succ 3 \succ 4$
- B: $2 \succ 3 \succ 4 \succ 1$

Sequence $\pi = ABBA \rightarrow \{14|23\}$.

Manipulation?

Example

2 agents, 4 objects:

- A: $1 \succ 2 \succ 3 \succ 4$
- B: $2 \succ 3 \succ 4 \succ 1$

Sequence $\pi = ABBA \to \{14|23\}$.

What if A knows B's preferences and acts maliciously?

Manipulation?

Example

2 agents, 4 objects:

- A: $1 \succ 2 \succ 3 \succ 4$
- B: $2 \succ 3 \succ 4 \succ 1$

Sequence $\pi = ABBA \rightarrow \{14|23\}$.

What if A knows B's preferences and acts maliciously?

She can manipulate by picking 2 instead of 1 at first step $\rightarrow \{12|34\}$.

More formally

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A **policy** π
- The agents have their own private preferences (which may or may not be additive) and use them for their **picking strategy**.

More formally

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A **policy** π
- The agents have their own private preferences (which may or may not be additive) and use them for their **picking strategy**.

The cheating agent (A) knows:

- the sequence
- her own (general) **picking strategy**
- the others' **picking strategy** (assumed to be **simple** and **deterministic** – as if each agent had an underlying linear order over the objects)

More formally

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A **policy** π
- The agents have their own private preferences (which may or may not be additive) and use them for their **picking strategy**.

The cheating agent (A) knows:

- the sequence
- her own (general) **picking strategy**
- the others' **picking strategy** (assumed to be **simple** and **deterministic** – as if each agent had an underlying linear order over the objects)

She wants:

- to get the best bundle she can get.

More formally

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A **policy** π
- The agents have their own private preferences (which may or may not be additive) and use them for their **picking strategy**.

The cheating agent (A) knows:

- the sequence
- her own (general) **picking strategy**
- the others' **picking strategy** (assumed to be **simple** and **deterministic** – as if each agent had an underlying linear order over the objects)

She wants:

- to get the best bundle she can get.

Her only possible cheating actions:

- choose at given steps **not** to pick her preferred objects.

Bad news...

(Folk?) theorem

The only strategyproof picking sequences are those made of contiguous blocks of agents (e.g. $A \ldots AB \ldots BC \ldots C$).

Bad news...

(Folk?) theorem

The only strategyproof picking sequences are those made of contiguous blocks of agents (e.g. $A \dots AB \dots BC \dots C$).

Is it hard to manipulate? (complexity as a barrier to manipulation)

First result

A: *"Can I get $\mathcal{S}$ for sure?"*

First result

A: *"Can I get $\mathcal{S}$ for sure?"*

Getting a subset for sure

We can answer to that constructively in polynomial time!

First result

A: *"Can I get $\mathcal{S}$ for sure?"*

Getting a subset for sure

We can answer to that constructively in polynomial time!

Idea:

- two agents: pick the objects in $\mathcal{S}$ in the same order of $\succ_B$
- more agents:
  - transform agents 2 to $m-1$ into a single (fake) agent
  - apply the algorithm for 2 agents

General manipulation problem

A: *"What is the best subset I can get?"*

General manipulation problem

A: *"What is the best subset I can get?"*

Idea: Greedily build the optimal achievable subset:

- Find the best object i such that $\{i\}$ is achievable;
- Find the best object j such that $\{i, j\}$ is achievable;
- ...

General manipulation problem

A: *"What is the best subset I can get?"*

Idea: Greedily build the optimal achievable subset:

- Find the best object i such that $\{i\}$ is achievable;
- Find the best object j such that $\{i, j\}$ is achievable;
- ...

Manipulation with additive preferences, two agents

If the manipulator has additive preferences, the optimal manipulation can be computed in polynomial time.

General manipulation problem

A: *"What is the best subset I can get?"*

Idea: Greedily build the optimal achievable subset:

- Find the best object i such that $\{i\}$ is achievable;
- Find the best object j such that $\{i, j\}$ is achievable;
- ...

Manipulation with additive preferences, two agents

If the manipulator has additive preferences, the optimal manipulation can be computed in polynomial time.

Only works **for two agents**!

General manipulation problem

[Aziz et al., 2017]

If the manipulator has additive preferences, the optimal manipulation problem is **NP**-complete.

(reduction from [3-SAT])

- Is there a manipulation that yields a better utility than the truthful report? $\rightsquigarrow$ **NP**-complete
- Not true anymore for binary utilities and (ordinal) responsive set extension.

Aziz, H., Bouveret, S., Lang, J., and Mackenzie, S. (2017).
Complexity of manipulating sequential allocation.
In *Proceedings of the 31st AAAI conference on Artificial Intelligence (AAAI'17).*

Coalitional Manipulation

Example

3 agents, 6 objects:

- A: $1 \succ 2 \succ 5 \succ 4 \succ 3 \succ 6$
- B: $1 \succ 3 \succ 5 \succ 2 \succ 4 \succ 6$
- C: $2 \succ 3 \succ 4 \succ 1 \succ 5 \succ 6$

Sequence $\pi = ABCABC \rightarrow \{15|34|26\}$.

Coalitional Manipulation

Example

3 agents, 6 objects:

- A: $1 \succ 2 \succ 5 \succ 4 \succ 3 \succ 6$
- B: $1 \succ 3 \succ 5 \succ 2 \succ 4 \succ 6$
- C: $2 \succ 3 \succ 4 \succ 1 \succ 5 \succ 6$

Sequence $\pi = ABCABC \rightarrow \{15|34|26\}$.

- If A and B manipulate alone, they cannot do better
- If they cooperate, they can get $\{12|35|46\}$, which is strictly better.

Coalitional Manipulation: Results

Three kinds of manipulation considered here:

- No post-allocation trade allowed between the manipulators
- Post-allocation exchange of goods allowed between the manipulators
- Post-allocation exchange of goods + side-payments allowed

Coalitional Manipulation: Results

Three kinds of manipulation considered here:

- No post-allocation trade allowed between the manipulators
- Post-allocation exchange of goods allowed between the manipulators
- Post-allocation exchange of goods + side-payments allowed

Results:

- No post-allocation trade allowed between the manipulators → **NP**-complete [PARTITION]
- Post-allocation exchange of goods allowed between the manipulators → **NP**-complete [PARTITION]
- Post-allocation exchange of goods + side-payments allowed → polynomial (comes down to manipulation by a single agent)

Everyone manipulates...

One manipulator

Everyone manipulates...

One manipulator $\rightarrow$ several manipulators (coalitional manipulation)

Everyone manipulates...

One manipulator → several manipulators (coalitional manipulation)
→ everyone (rational, self-interested) manipulates?

Everyone manipulates...

One manipulator → several manipulators (coalitional manipulation)
→ everyone (rational, self-interested) manipulates?

Game Theory (Subgame Perfect Nash Equilibrium)

Everyone manipulates...

One manipulator → several manipulators (coalitional manipulation) → everyone (rational, self-interested) manipulates?

Game Theory (Subgame Perfect Nash Equilibrium)

- Two agents and additive utilities, precise characterization of the result of every SPNE $((rev(\succ_2), rev(\succ_1), rev(\pi)))$ [Kalinowski et al., 2013, Kohler and Chandrasekaran, 1971].
- Unbounded number of agents: **PSPACE**-hard [Kalinowski et al., 2013].

Kalinowski, T., Narodytska, N., Walsh, T., and Xia, L. (2013).
Strategic behavior when allocating indivisible goods sequentially.
In *Proceedings of AAAI'13*.

Kohler, D. A. and Chandrasekaran, R. (1971).
A class of sequential games.
Operations Research, 19(2):270–277.

Picking sequences, fairness, efficiency

A simple yet powerful allocation mechanism

Fair division with additive preferences

In this section, we focus on **fair division with additive preferences**:

Fair division with additive preferences

In this section, we focus on **fair division with additive preferences**:

- a finite set of objects $\mathcal{O} = \{1, \ldots, m\}$

o_1 o_2 o_3

Fair division with additive preferences

In this section, we focus on **fair division with additive preferences**:

- a finite set of objects $\mathcal{O} = \{1, \ldots, m\}$
- a finite set of agents $\mathcal{A} = \{1, \ldots, n\}$

	o_1	o_2	o_3
agent 1			
agent 2			

Fair division with additive preferences

In this section, we focus on **fair division with additive preferences**:

- a finite set of objects $\mathcal{O} = \{1, \ldots, m\}$
- a finite set of agents $\mathcal{A} = \{1, \ldots, n\}$
- Additive preferences: $\rightarrow w_i(o_j)$ (agent a, object i_j) .

	o_1	o_2	o_3
agent 1	5	4	2
agent 2	4	1	6

Fair division with additive preferences

In this section, we focus on **fair division with additive preferences**:

- a finite set of objects $\mathcal{O} = \{1, \ldots, m\}$
- a finite set of agents $\mathcal{A} = \{1, \ldots, n\}$
- Additive preferences: $\rightarrow w_i(o_j)$ (agent a, object i_j)
  $\rightarrow u_i(\mathcal{X}) = \sum_{o_j \in \mathcal{X}} w_i(o_j)$.

	o_1	o_2	o_3
agent 1	5	4	2
agent 2	4	1	6

$u_2(\{2, 3\}) = 1 + 6 = 7$

Fair division with additive preferences

In this section, we focus on **fair division with additive preferences**:

- a finite set of objects $\mathcal{O} = \{1, \dots, m\}$
- a finite set of agents $\mathcal{A} = \{1, \dots, n\}$
- Additive preferences: $\rightarrow w_i(o_j)$ (agent a, object i_j)
  $\rightarrow u_i(\mathcal{X}) = \sum_{o_j \in \mathcal{X}} w_i(o_j)$.

	o_1	o_2	o_3
agent 1	5	4	2
agent 2	4	1	6

Fair division with additive preferences

In this this section, we focus on **fair division with additive preferences**:

- a finite set of objects $\mathcal{O} = \{1, \ldots, m\}$
- a finite set of agents $\mathcal{A} = \{1, \ldots, n\}$
- Additive preferences: $\rightarrow w_i(o_j)$ (agent a, object i_j)
  $\rightarrow u_i(\mathcal{X}) = \sum_{o_j \in \mathcal{X}} w_i(o_j)$.

	o_1	o_2	o_3
agent 1	5	4	2
agent 2	4	1	6

$\overrightarrow{\pi} = \langle \{1\}, \{2, 3\} \rangle$

- $u_1(\overrightarrow{\pi}) = 5$
- $u_2(\overrightarrow{\pi}) = 7$

Fair division with additive preferences

In this this section, we focus on **fair division with additive preferences**:

- a finite set of objects $\mathcal{O} = \{1, \ldots, m\}$
- a finite set of agents $\mathcal{A} = \{1, \ldots, n\}$
- Additive preferences: $\rightarrow w_i(o_j)$ (agent a, object i_j)
  $\rightarrow u_i(\mathcal{X}) = \sum_{o_j \in \mathcal{X}} w_i(o_j)$.

	o_1	o_2	o_3
agent 1	5	4	2
agent 2	4	1	6

$\overrightarrow{\pi} = \langle \{1\}, \{2, 3\} \rangle$

- $u_1(\overrightarrow{\pi}) = 5$
- $u_2(\overrightarrow{\pi}) = 7$

Pareto-efficiency

Pareto-efficiency: we cannot improve the utility of an agent without decreasing the utility of another one.

Pareto-efficiency

Pareto-efficiency: we cannot improve the utility of an agent without decreasing the utility of another one.

Picking sequences: a form of "local efficiency". At her turn, an agent picks the best item for her.

Pareto-efficiency

Pareto-efficiency: we cannot improve the utility of an agent without decreasing the utility of another one.

Picking sequences: a form of “local efficiency”. At her turn, an agent picks the best item for her.

Is there a link between Pareto-efficiency and picking sequences?

Pareto-efficiency and picking sequences

Yes, there is a link between Pareto-efficiency and picking sequences...

Proposition [Brams and King, 2005]

π Pareto-efficient $\Leftrightarrow$ sequenceable.

(here π Pareto-efficient $\Leftrightarrow$ for all π', there exists one utility function u compatible with $\succ$ such that π' does not Pareto-dominate π for u.)

Brams, S. and King, D. (2005).
Efficient fair division: Help the worst off or avoid envy ?
Rationality and Society, 17:387–421.

Pareto-efficiency and picking sequences

Yes, there is a link between Pareto-efficiency and picking sequences...

Proposition [Brams and King, 2005]

π Pareto-efficient $\Leftrightarrow$ sequenceable.

(here π Pareto-efficient $\Leftrightarrow$ for all π', there exists one utility function u compatible with $\succ$ such that π' does not Pareto-dominate π for u.)

Brams, S. and King, D. (2005).
Efficient fair division: Help the worst off or avoid envy ?
Rationality and Society, 17:387–421.

Possible Pareto-efficiency $\rightsquigarrow$ very weak.

Pareto-efficiency and picking sequences

In general, π Pareto-efficient $\not\Leftrightarrow$ sequenceable.

	o_1	o_2	o_3
agent 1	4	2	5
agent 2	2	1	8

Pareto-efficiency and picking sequences

In general, π Pareto-efficient $\not\Leftarrow$ sequenceable.

	o_1	o_2	o_3
agent 1	4	2	5
agent 2	2	1	8

Observation: $\langle\{3\},\{1,2\}\rangle$ is sequenceable but not Pareto-efficient.

Pareto-efficiency and picking sequences

In general, π Pareto-efficient $\not\Leftrightarrow$ sequenceable.

	o_1	o_2	o_3
agent 1	**4**	**2**	5
agent 2	2	1	**8**

Observation: $\langle \{3\}, \{1, 2\} \rangle$ is sequenceable but not Pareto-efficient.

Why? It is dominated by $\langle \{1\}, \{2, 3\} \rangle$

Pareto-efficiency and picking sequences

However, there is the following result:

Proposition [Aziz et al., 2016]

Every Pareto-efficient allocation is sequenceable.

Aziz, H., Biró, P., Lang, J., Lesca, J., and Monnot, J. (2016).
Optimal reallocation under additive and ordinal preferences.
In *Proceedings of the 15th International Conference on Autonomous Agents and Multiagent Systems (AAMAS'16)*, pages 402–410, Richland, SC. International Foundation for Autonomous Agents and Multiagent Systems.

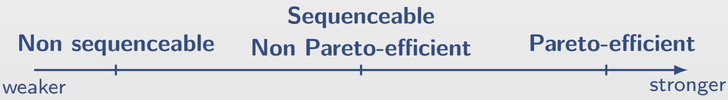
A scale of efficiency

Every Pareto-efficient allocation is sequenceable.
But Pareto-efficiency $\not\Leftarrow$ sequenceability.

A scale of efficiency

Every Pareto-efficient allocation is sequenceable.
But Pareto-efficiency $\not\Leftarrow$ sequenceability.

A scale of efficiency...



Trading as an allocation procedure

Now let's speak about another allocation procedure...

Trading as an allocation procedure

Now let's speak about another allocation procedure...

- Start from an initial allocation

Trading as an allocation procedure

Now let's speak about another allocation procedure...

- Start from an initial allocation
- Let the agents **negotiate**

Trading as an allocation procedure

Now let's speak about another allocation procedure...

- Start from an initial allocation
- Let the agents **negotiate**

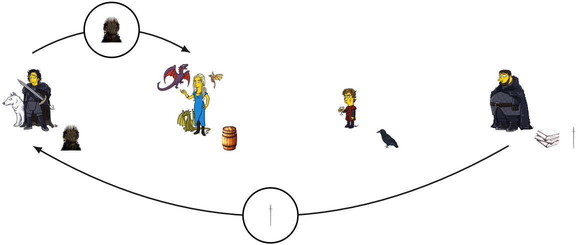
A particular kind of negotiation scheme: **trading cycles**

Sandholm, T. W. (1998).
Contract types for satisficing task allocation: I. theoretical results.
In Sen, S., editor, *Proceedings of the AAAI Spring Symposium: Satisficing Models*, pages 68–75, Menlo Park, California. AAAI Press.

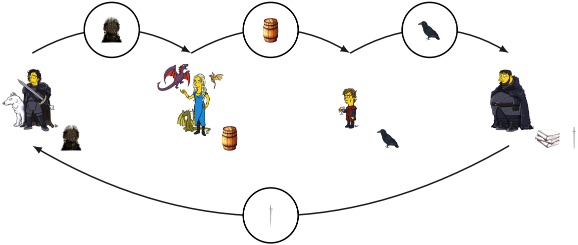
Shapley, L. and Scarf, H. (1974).
On cores and indivisibility.
Journal of mathematical economics, 1(1):23–37.

Trading cycles

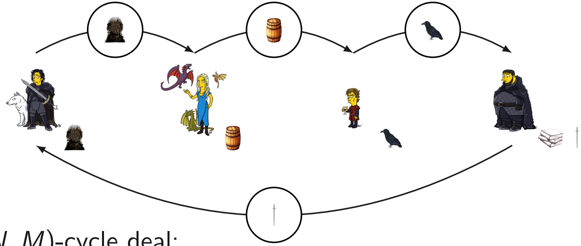
Trading cycles



Trading cycles



Trading cycles

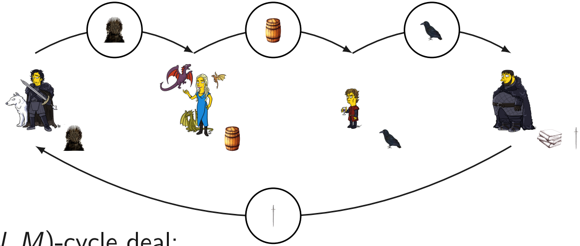


(N, M)-cycle deal:

- N: cycle length
- M: max number of objects involved in each trade

(in the example above, $N = 4$ and $M = 1$)

Trading cycles



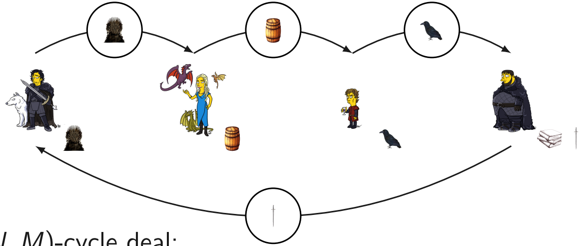
(N, M)-cycle deal:

- N: cycle length
- M: max number of objects involved in each trade

(in the example above, $N = 4$ and $M = 1$)

Interesting deals: **improving deals**.

Trading cycles



(N, M)-cycle deal:

- N: cycle length
- M: max number of objects involved in each trade

(in the example above, $N = 4$ and $M = 1$)

Interesting deals: **improving deals**.

Notion of efficiency: **cycle-deal optimality**.

Trading cycles and picking sequences

Notion of efficiency: **cycle-deal optimality**.

Is there a link with Pareto-efficiency and picking sequences?

Trading cycles and picking sequences

Notion of efficiency: **cycle-deal optimality**.

Is there a link with Pareto-efficiency and picking sequences?

Proposition [Beynier et al., 2019]

$\overrightarrow{\pi}$ $(n, 1)$-cycle optimal $\Leftrightarrow$ $\overrightarrow{\pi}$ sequenceable.

Beynier, A., Bouveret, S., Lemaître, M., Maudet, N., Rey, S., and Shams, P. (2019).
Efficiency, sequenceability and deal-optimality in fair division of indivisible goods.
In *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems (AAMAS'19)*. IFAAMAS.

Approximating Envy-freeness

We have seen the link between picking sequences and efficiency. Now, what about **fairness**?

Approximating Envy-freeness

We have seen the link between picking sequences and efficiency. Now, what about **fairness**?

Envy-freeness: everyone thinks her share is better than any other share.

Approximating Envy-freeness

We have seen the link between picking sequences and efficiency. Now, what about **fairness**?

Envy-freeness: everyone thinks her share is better than any other share.

Nice property, but cannot always be satisfied. $\rightsquigarrow$ several relaxations proposed:

- envy-minimization (for several definitions of the quantity of envy);
- envy-freeness up to one good [Budish, 2011]: "I might envy some other agent, but if we remove just one good from the share of this agent, then I do not envy her anymore."

Budish, E. (2011).
The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes.
Journal of Political Economy, 119(6).

Envy-freeness Up to One Good

Proposition [Budish, 2011, Caragiannis et al., 2016]

Any allocation obtained by the round-robin picking sequence is envy-free up to one good.

Budish, E. (2011).
The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes.
Journal of Political Economy, 119(6).

Caragiannis, I., Kurokawa, D., Moulin, H., Procaccia, A. D., Shah, N., and Wang, J. (2016).
The unreasonable fairness of maximum nash welfare.
In *Proceedings of the ACM Conference on Electronic Commerce (EC'16)*.

CEEI

Another interesting property:

Proposition [Beynier et al., 2019]

A Competitive Equilibrium from Equal Income can be non Pareto-efficient, but is always sequenceable.

Beynier, A., Bouveret, S., Lemaître, M., Maudet, N., Rey, S., and Shams, P. (2019).
Efficiency, sequenceability and deal-optimality in fair division of indivisible goods.
In *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems (AAMAS'19)*. IFAAMAS.

Price of elicitation-freeness

Another way to ensure fairness and efficiency: maximize a collective utility (social choice) function.

Price of elicitation-freeness

Another way to ensure fairness and efficiency: maximize a collective utility (social choice) function.

Picking sequences: arguably a very simple (and natural) protocol...

Price of elicitation-freeness

Another way to ensure fairness and efficiency: maximize a collective utility (social choice) function.

Picking sequences: arguably a very simple (and natural) protocol…

…but obviously suboptimal

Price of elicitation-freeness

Another way to ensure fairness and efficiency: maximize a collective utility (social choice) function.

Picking sequences: arguably a very simple (and natural) protocol...

...but obviously suboptimal

Question: *What is the loss in social welfare we incur by using such a simple protocol instead of computing the optimal allocation?*

Here we focus on **regular sequences** σ of the kind $(1 \dots n)^*$ (round-robin), but our results are similar for alternating sequences like $(1 \dots nn \dots 1)^*$.

Price of elicitation-freeness

Another way to ensure fairness and efficiency: maximize a collective utility (social choice) function.

Picking sequences: arguably a very simple (and natural) protocol...

...but obviously suboptimal

Question: *What is the loss in social welfare we incur by using such a simple protocol instead of computing the optimal allocation?*

Here we focus on **regular sequences** σ of the kind $(1 \dots n)^*$ (round-robin), but our results are similar for alternating sequences like $(1 \dots nn \dots 1)^*$.

More formally:

- **Multiplicative** Price of Elicitation-Freeness: worst case ratio $u_c^{\text{opt}}/u_c(\sigma)$, for a sequence σ.
- **Additive** Price of Elicitation-Freeness: worst case difference $u_c^{\text{opt}} - u_c(\sigma)$, for a sequence σ.

Some theoretical bounds for Borda

For **classical utilitarianism** ($\sum_i u_i(\pi)$):

$$1 + \frac{n-1}{m} + \Theta(\frac{1}{m^2}) \leq MPEF \leq 2 - \frac{1}{n} + \Theta(\frac{1}{m^2}), \text{ when } m \to +\infty.$$

Some theoretical bounds for Borda

For **classical utilitarianism** ($\sum_i u_i(\pi)$):

$$1 + \frac{n-1}{m} + \Theta(\frac{1}{m^2}) \leq MPEF \leq 2 - \frac{1}{n} + \Theta(\frac{1}{m^2}), \text{ when } m \to +\infty.$$

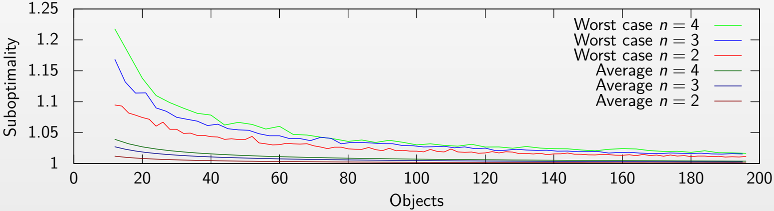
For **egalitarianism** ($\min_i u_i(\pi)$):

$$MPEF \leq 2 - \frac{1}{n} + \Theta(\frac{1}{m^2}), \text{ when } m \to +\infty.$$

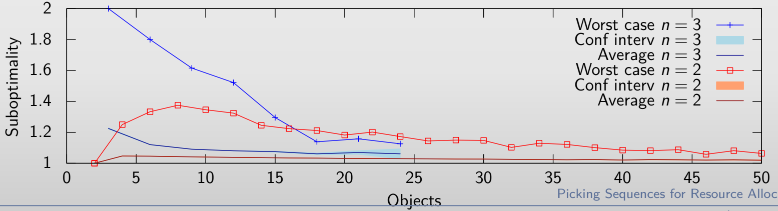
Baumeister, D., Bouveret, S., Lang, J., Nguyen, N.-T., Nguyen, T., Rothe, J., and Saffidine, A. (2017).
Positional scoring-based allocation of indivisible goods.
Autonomous Agents and Multi-Agent Systems, 31(3):628–655.

Some experimental results for Borda

For **classical utilitarianism** ($\sum_i u_i(\pi)$):



For **egalitarianism** ($\min_i u_i(\pi)$):



Conclusion

A take-away message?

Conclusion

- A simple and intuitive sequential allocation procedure (actually already known in sparse litterature)
- Finding the best policy?
    - Full Correlation case well understood
    - Full Independence: partial results
- Strategical issues: individual and coalitional manipulation; game-theoretic approach

Very nice properties...

A simple protocol, but with nice features:

- "locally" efficient
- efficient with respect to cycle deals
- guarantees envy-freeness up to one good
- gives good approximation of social welfare
- also gives good approximation of other fairness properties (*e.g.* max-min share)

Thank you

Want to read more?

http://recherche.noiraudes.net/en/picking.php

Pictures (shamefully) borrowed without permission from ADN (https://drawthesimpsons.tumblr.com/)

Want to do a PhD or a postdoc in Grenoble?

Contact me!

`sylvain.bouveret@imag.fr`