



Fair division of indivisible goods under risk

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Classical fair division

Fair division of indivisible goods...

We have :

- a finite set of **objects** $\mathcal{O} = \{1, \dots, l\}$
- a finite set of **agents** $\mathcal{A} = \{1, \dots, n\}$ having some **preferences** on the set of objects they may receive

We want :

- an allocation $\pi : \mathcal{A} \rightarrow 2^{\mathcal{O}}$
- such that $\pi_i \cap \pi_j = \emptyset$ if $i \neq j$
- and which takes into account the agents' preferences



Examples

A toy-example :



- A set of bottles of wine to share. . .
- **Objects** : bottles of wine
- **Agents** : wine amateurs



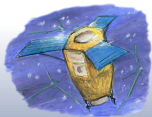
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A more realistic example :



- A co-funded Earth-observing satellite to operate. . .
- **Agents** : the countries that have co-funded the satellite
- **Objects** : observation requests posted by the agents



Centralized allocation

A classical way to solve the problem :

- Ask the agents to give a score (weight, utility. . .) $w(o)$ to each object o
- Consider that they have **additive** preferences $\rightarrow u(\pi) = \sum_{o \in \pi} w(o)$
- Find an allocation that maximizes $\min_{i \in \mathcal{A}} u(\pi(i))$ (egalitarian solution [Rawls, 1971])



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Example : 3 objects $\{j_1, j_2, j_3\}$, 2 agents $\{i_1, i_2\}$

Preferences :

	j_1	j_2	j_3
i_1	5	4	2
i_2	4	1	4

$$\pi = \langle \{j_1, j_2\}, \{j_3\} \rangle \rightarrow uc(\pi) = \min(4 + 5, 4) = 4$$

$$\pi' = \langle \{j_1\}, \{j_2, j_3\} \rangle \rightarrow uc(\pi') = \min(5, 4 + 1) = 5$$



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The Santa-Claus problem [Bansal and Sviridenko, 2006]



Bansal, N. and Sviridenko, M. (2006).

The santa claus problem.

In *Proceedings of the thirty-eighth annual ACM symposium on Theory of computing*, pages 31–40. ACM.



Adding uncertainty

Now, we might be unsure of the quality of the objects when they are allocated.



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We assume that :

- each object can be in **two possible states** : good or bad (bad = utility 0)
- each object o has a probability $p(o)$ to be good
- these probabilities are independent



Resource allocation under risk

Resource allocation problem

A tuple $(\mathcal{A}, \mathcal{O}, \mathbf{W}, \mathbf{p})$ with :

- $\mathcal{A} = \{1, \dots, n\}$ a set of agents
- $\mathcal{O} = \{1, \dots, l\}$ a set of objects
- $\mathbf{W} \in \mathcal{M}_{n,l}(\mathbb{R}^+)$ a matrix of weights (given by agents to objects)
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Notations :

- \mathcal{S} : the set of 2^l states of the world
- $good(s) \subseteq \mathcal{O}$: the set of objects in good states in $s \in \mathcal{S}$
- $u_{i,s}(\pi)$ the utility of agent i in s with allocation π .



The model

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$$\forall j, p_j = 0.5$$



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Profiles :

$$\pi \longrightarrow \begin{bmatrix} 0 & 0 & 4 & 4 & 5 & 5 & 9 & 9 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \end{bmatrix}$$

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$\downarrow \text{min}$
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The model

Timing effect [Myerson, 1981]



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Ex-ante collective utility

$$acu(\pi) = \min_{i \in \mathcal{A}} \left(\sum_{s \in \mathcal{S}} \Pr(s) \cdot u_{i,s}(\pi) \right)$$



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Proposition

$$\forall \pi, \quad acu(\pi) \geq pcu(\pi)$$



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The $\tilde{w}_{ij} = p_j w_{ij}$ can be pre-computed in linear time



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Risk-free equivalent problem

$$(\mathcal{A}, \mathcal{O}, \mathbf{W}, \mathbf{p}) \iff (\mathcal{A}, \mathcal{O}, \tilde{\mathbf{W}})$$



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- we cannot pre-compute expected utilities
- we must enumerate all the states of the world



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Branch on objects (good / bad states) → possible heuristics : small shares first.



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•

j_3

$pcu(\pi) =$

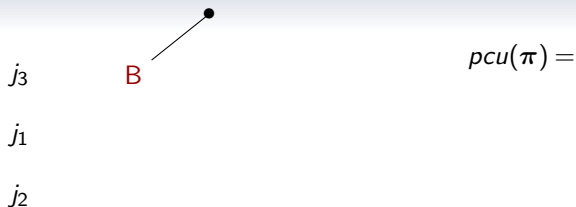
j_1

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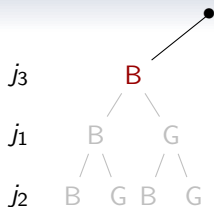


$$p = 0.5$$



Example

2 agents $\{i_1, i_2\}$, 3 objects $\{j_1, j_2, j_3\}$, $\pi = \langle \{j_1, j_2\}, \{j_3\} \rangle$



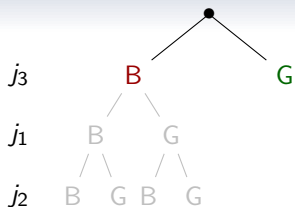
$$pcu(\pi) = 0.5 \times \min\{?, 0\}$$

$$p = 0.5$$



Example

2 agents $\{i_1, i_2\}$, 3 objects $\{j_1, j_2, j_3\}$, $\pi = \langle \{j_1, j_2\}, \{j_3\} \rangle$



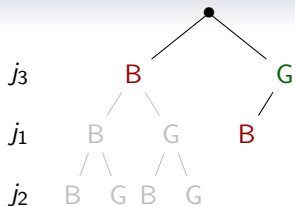
$$p = 0.5$$

$$pcu(\pi) = 0.5 \times \min\{?, 0\}$$



Example

2 agents $\{i_1, i_2\}$, 3 objects $\{j_1, j_2, j_3\}$, $\pi = \langle \{j_1, j_2\}, \{j_3\} \rangle$



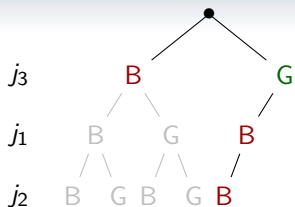
$$p = 0.25$$

$$pcu(\pi) = 0.5 \times \min\{?, 0\}$$



Example

2 agents $\{i_1, i_2\}$, 3 objects $\{j_1, j_2, j_3\}$, $\pi = \langle \{j_1, j_2\}, \{j_3\} \rangle$



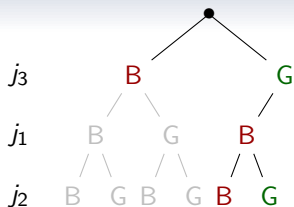
$$p = 0.125$$

$$\begin{aligned} p_{cu}(\pi) = & \\ & 0.5 \times \min\{?, 0\} \\ & + 0.125 \times \min\{0, 4\} \end{aligned}$$



Example

2 agents $\{i_1, i_2\}$, 3 objects $\{j_1, j_2, j_3\}$, $\pi = \langle \{j_1, j_2\}, \{j_3\} \rangle$

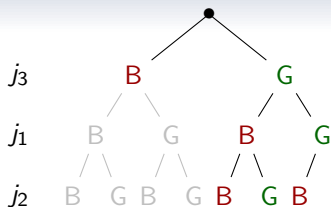


$$\begin{aligned}pcu(\pi) = & 0.5 \times \min\{?, 0\} \\ & + 0.125 \times \min\{0, 4\} \\ & + 0.125 \times \min\{4, 4\}\end{aligned}$$



Example

2 agents $\{i_1, i_2\}$, 3 objects $\{j_1, j_2, j_3\}$, $\pi = \langle \{j_1, j_2\}, \{j_3\} \rangle$

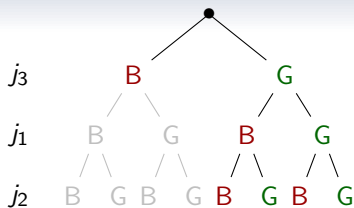


$$\begin{aligned} p_{cu}(\pi) = & 0.5 \times \min\{?, 0\} \\ & + 0.125 \times \min\{0, 4\} \\ & + 0.125 \times \min\{4, 4\} \\ & + 0.125 \times \min\{5, 4\} \end{aligned}$$



Example

2 agents $\{i_1, i_2\}$, 3 objects $\{j_1, j_2, j_3\}$, $\pi = \langle \{j_1, j_2\}, \{j_3\} \rangle$

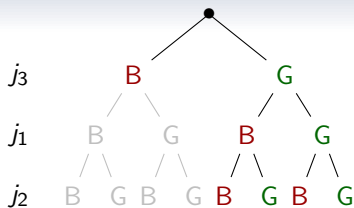


$$\begin{aligned}
 pcu(\pi) = & 0.5 \times \min\{?, 0\} \\
 & + 0.125 \times \min\{0, 4\} \\
 & + 0.125 \times \min\{4, 4\} \\
 & + 0.125 \times \min\{5, 4\} \\
 & + 0.125 \times \min\{9, 4\}
 \end{aligned}$$



Example

2 agents $\{i_1, i_2\}$, 3 objects $\{j_1, j_2, j_3\}$, $\pi = \langle \{j_1, j_2\}, \{j_3\} \rangle$



$$\begin{aligned}
 pcu(\pi) &= \\
 & 0.5 \times \min\{?, 0\} \\
 & + 0.125 \times \min\{0, 4\} \\
 & + 0.125 \times \min\{4, 4\} \\
 & + 0.125 \times \min\{5, 4\} \\
 & + 0.125 \times \min\{9, 4\} \\
 & = 1.5
 \end{aligned}$$



Branch and Bound

- **Variables** : objects
- **Question** : to whom is it allocated ?
- **Heuristics** : give to the poorest agent the object she prefers
- **Possible cuts** : *ex-ante* collective utility $\overline{ac\bar{u}}$ of a virtual allocation which gives to all agents all the still unallocated objects
- The *ex-post* utility is computed only at each leaf of the search tree



Mixed utility

The *ex-post* utility is computed only at each leaf of the search tree. . .



Mixed utility

The *ex-post* utility is computed only at each leaf of the search tree. . .
. . . but it is still very expensive. . .



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Wouldn't it be possible to define and use a better approximation than \overline{acu} ?



Mixed utility

The *ex-post* utility is computed only at each leaf of the search tree . . .
. . . but it is still very expensive . . .

Wouldn't it be possible to define and use a better approximation than \overline{acu} ?

Idea of the **mixed utility** $mcu(\pi, \Omega)$:

- a set Ω of objects which are computed *ex-ante*
- objects from $\mathcal{O} \setminus \Omega$ are still computed *ex-post*
- we still use \overline{acu} as an upper bound in the search tree
- we use $mcu(\pi, \Omega)$ at each leaf to avoid unnecessary *ex-post* computations



Some results

n	l	(a)	(b)	(c)	(d)
5	≤ 9	100	100	100	100
5	10	49	52	89	100
5	11	1	1	10	52
5	≥ 12	0	0	0	0

n	l	(a)	(b)	(c)	(d)
7	≤ 8	100	100	100	100
7	9	27	47	100	100
7	10	0	1	19	32
7	≥ 11	0	0	0	0

FIGURE: Number of instances solved in 30 seconds (over 100 instances)

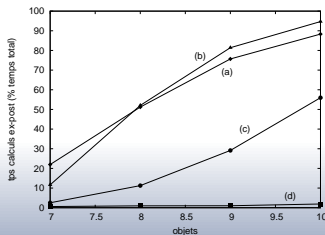
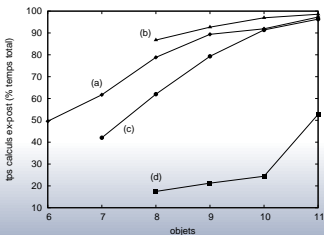


FIGURE: Percentage of the total time used for *ex-post* computations for 5 and 7 agents (average over 100 instances)



An approached algorithm

- **Approximate computation** of pcu :
 - with mixed collective utility
 - with a Monte-Carlo procedure



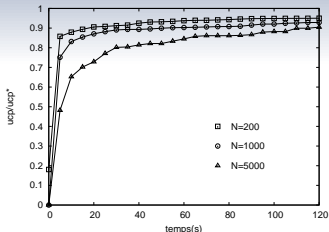
An approached algorithm

- **Approximate computation** of pcu :
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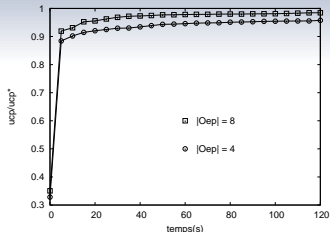
- Customized **greedy stochastic algorithm** [Bresina, 1996] :
 - "best" solutions stored
 - exact evaluation of stored solutions



Some results



(a) **Monte Carlo** approximation, for different numbers of draws



(b) **Mixed utility** based approximation, for different sizes of Ω

FIGURE: Evolution of the best solution (average over 100 instances with 5 agents and 12 objects)



Summary

- **The model** : a Santa-Claus problem under risk
 - two possible states for each object
 - each object is in good state with a given probability
 - two possible egalitarian collective utility functions : *ex-ante* and *ex-post*
- **Ex-ante** case can be reduced to risk-free allocation
- **Ex-post** optimization :
 - a (supposed) quite harder problem
 - a branch-and-bound algorithm with mixed utility
 - some incomplete methods



Future work

- On this problem :
 - Missing complexity result
 - How to choose the objects in Ω for mixed utility?
 - Better algorithms?
- Other problems :
 - Matching problems ($l \leq n$) with other CUF
 - Relaxing probabilistic independence (Bayesian networks)
 - More possible states for each object



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