Fairness Criteria for Fair Division of Indivisible Goods

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A fair division problem...

You have:

- a finite set of **objects** $\mathcal{O} = \{1, \ldots, m\}$
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How would you allocate the objects to the agents so as to be as ***fair*** *as possible?*

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*How would you allocate the objects to the agents so as to be as **fair** as possible?*

More precisely, you want:

- an allocation $\overrightarrow{\pi} : \mathcal{A} \to 2^{\mathcal{O}}$
- such that $\pi_i \cap \pi_j = \emptyset$ if $i \neq j$ (preemption),
- $\bigcup_{i \in \mathcal{A}} \pi_i = \mathcal{O}$ (no free-disposal),
- and which takes into account the agents' preferences

An ubiquitous problem

- Allocation of courses (or practical works) to students
- Allocation of take-off and landing slots in airports
- Allocation of tasks to workers
- Allocation of jobs to machines
- Allocation satellite resources

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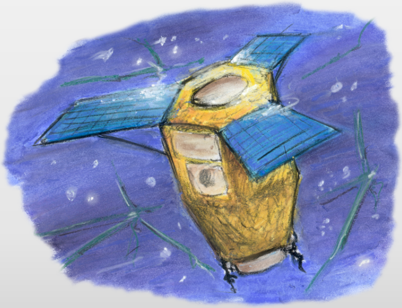

A rich problem

Fair Division

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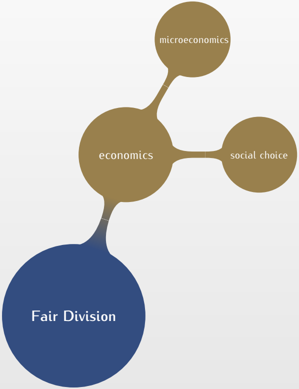

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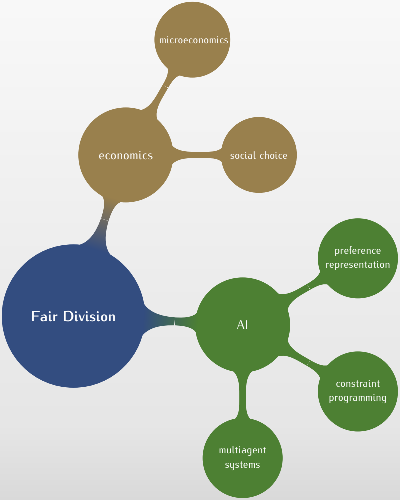

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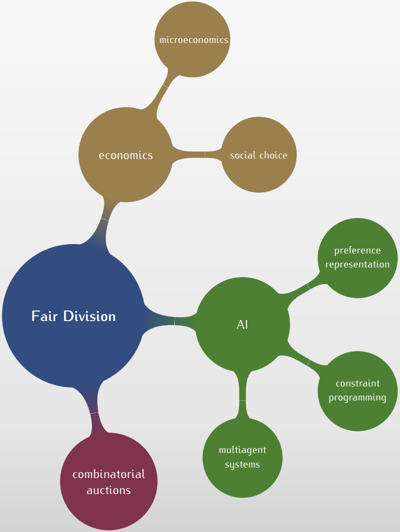

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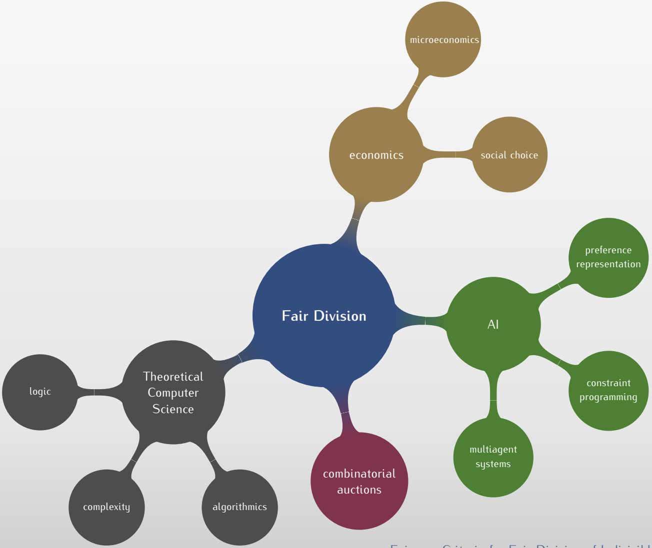

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Computational social choice

Computational Social Choice (COMSOC)

COMSOC $\approx$ Social Choice $\cap$ Computer Science

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1. Use techniques from economics to solve problems in IT (network sharing, job allocation...)
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- Fair division (indivisible goods or cake-cutting)
- Voting
- Coalition formation / hedonic games
- Judgment aggregation...

Back to our fair division problem

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Centralized allocation

A classical way to solve the problem:

- Ask each agent i to give a score (weight, utility...) $w_i(o)$ to each object o
- Consider all the agents have **additive** preferences

$$\rightarrow u_i(\pi) = \sum_{o \in \pi} w_i(o)$$

- Find an allocation $\overrightarrow{\pi}$ that:

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  $\rightarrow u_i(\pi) = \sum_{o \in \pi} w_i(o)$
- Find an allocation $\overrightarrow{\pi}$ that:

1 | maximizes the collective utility defined by a **collective utility function**,

e.g. $uc(\overrightarrow{\pi}) = \min_{i \in \mathcal{A}} u_i(\pi_i)$ – egalitarian solution [Bansal and Sviridenko, 2006]

2 | or satisfies a given **fairness criterion**,

e.g. $u_i(\pi_i) \geq u_i(\pi_j)$ for all agents i, j – envy-freeness [Lipton et al., 2004].

Bansal, N. and Sviridenko, M. (2006).
The Santa Claus problem.
In *Proceedings of STOC'06*. ACM.

Lipton, R., Markakis, E., Mossel, E., and Saberi, A. (2004).
On approximately fair allocations of divisible goods.
In *Proceedings of EC'04*.

Example

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$\overrightarrow{\pi} = \langle \{1\}, \{2, 3\} \rangle \rightarrow uc(\overrightarrow{\pi}) = \min(5, 6 + 1) = 5$

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$\overrightarrow{\pi}$ is **not** envy-free (agent 1 envies agent 2)

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$\overrightarrow{\pi}'$ is envy-free.

Fairness properties

In this work, we consider the 2^{nd} approach: choose a **fairness property**, and find an allocation that satisfies it.

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1 such an allocation does not always exist
$\rightarrow$ *e.g.* 2 agents, 1 object: no envy-free allocation exists

2 many such allocations can exist

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1. such an allocation does not always exist
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2. many such allocations can exist

Idea: consider several fairness properties, and try to satisfy the most demanding one.
In this work we consider five such properties.

Five fairness criteria

Envy-freeness

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Known facts:

- An envy-free allocation may not exist.
- Deciding whether an allocation is envy-free is easy (quadratic time).
- Deciding whether an instance (agents, objects, preferences) has an envy-free allocation is hard – **NP**-complete [Lipton et al., 2004].

Lipton, R., Markakis, E., Mossel, E., and Saberi, A. (2004).
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Proportional fair share

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Proportional fair share (PFS):

- Initially defined [Steinhaus, 1948] for continuous fair division (*cake-cutting*)
- **Idea:** each agent is "entitled" to at least the nth of the entire resource

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Proportional fair share

The **proportional fair share** of an agent i is equal to:

$$u_i^{\text{PFS}} \stackrel{\text{def}}{=} \frac{u_i(\mathcal{O})}{n} = \sum_{o \in \mathcal{O}} \frac{w_i(o)}{n}$$

An allocation $\overrightarrow{\pi}$ satisfies **(proportional) fair share** if every agent gets at least her fair share.

Proportional fair share: facts

Easy or known facts:

- Deciding whether an allocation satisfies proportional fair share (PFS) is easy (linear time).
- For a given instance, there may be no allocation satisfying PFS
  $\rightarrow$ *e.g.* 2 agents, 1 object
- This is not true for cake-cutting (divisible resource)
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- Deciding whether an instance has an allocation satisfying PFS is hard even for 2 agents – **NP**-complete [PARTITION].
- $\vec{\pi}$ is envy-free $\Rightarrow$ $\vec{\pi}$ satisfies PFS.[1]

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Max-min fair share (MFS):

- Introduced recently [Budish, 2011]; not so much studied so far.
- **Idea:** in the **cake-cutting** case, PFS = the best share an agent can hopefully get for sure in a *"I cut, you choose (I choose last)"* game.
- Same game for indivisible goods → MFS.

Budish, E. (2011).
The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes.
Journal of Political Economy, 119(6).

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Max-min fair share: examples

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Example: 2 agents, 1 object.

$u_1^{\text{MFS}} = u_2^{\text{MFS}} = 0 \rightarrow$ every allocation satisfies MFS!

Not very satisfactory, but can we do much better?

Max-min fair share: properties

Facts:

- Computing u_i^{MFS} for a given agent is hard $\rightarrow$ **NP**-complete [PARTITION]
- Hence, deciding whether an allocation satisfies MFS is probably also hard (**coNP**-complete?)
- $\overrightarrow{\pi}$ satisfies PFS $\Rightarrow$ $\overrightarrow{\pi}$ satisfies MFS.

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- Proved for **special cases** (2 agents, matching,...), even **very general ones** (scoring functions...)
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Max-min fair share: conjecture

Conjecture

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FALSE!

- Proved for **special cases** (2 agents, matching,...), even **very general ones** (scoring functions...)
- No counterexample found on **thousands** of **random** instances.

The conjecture has been proved **false** by Procaccia and Wang using a **very tricky** counterexample (they also prove that 2/3 approximation is always achievable).

Procaccia, A. D. and Wang, J. (2014).
Fair enough: Guaranteeing approximate maximin shares.
In *Proc. 14th ACM Conference on Economics and Computation (EC'14)*.
forthcoming.

Max-min fair share: new facts

Since Procaccia and Wang's work...

Max-min fair share: new facts

Since Procaccia and Wang's work...

- Let $n \geq 3$:
  - if $m \leq n + 4$ an MMS allocation exists for sure [Kurokawa et al., 2015]
  - if $m \geq 3n + 4$ we can find an instance without MMS allocation [Kurokawa et al., 2015]
  - in between?
- 2/3-approximation in Polynomial time [Amanatidis et al., 2015] (7/8 for the 3-agent case)
- An MMS allocation exists with (theoretical) very high probability [Amanatidis et al., 2015, Kurokawa et al., 2015]

Amanatidis, G., Markakis, E., Nikzad, A., and Saberi, A. (2015).
Approximation algorithms for computing maximin share allocations.
In *ICALP (1)*, volume 9134 of *Lecture Notes in Computer Science*, pages 39–51. Springer.

Kurokawa, D., Procaccia, A. D., and Wang, J. (2015).
When can the maximin share guarantee be guaranteed?
Technical report, Carnegie Mellon University.

Max-min fair share and egalitarian allocation

"Max-min fair share" sounds like "max-min optimality"...

Idea: Use the egalitarian approach to compute

$$\widehat{\vec{\pi}} = \text{argmax}_{\vec{\pi}}(min_{i \in \mathcal{A}} u_i(\pi_i))$$

Santa-Claus problem [Bansal and Sviridenko, 2006] (connection to maximum makespan minimization in job scheduling on multiple machines), and it will give an MFS allocation

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Bad luck: there exist instances with MMS allocations, for which $\{$MMS allocations $\cap$ (lexi-)min optimal allocations$\} = \emptyset$.

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The Santa Claus problem.
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Computing a MFS allocation

- 2/3-approximation in Polynomial time [Amanatidis et al., 2015] (7/8 for the 3-agent case)
- **Open question:** complexity of deciding whether an instance is MFS?
- **Open question:** computing an MFS allocation (when there is one...) efficiently (Santa-Claus may help but is not the answer)

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The **min-max fair share** of an agent i is equal to:

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An allocation $\overrightarrow{\pi}$ satisfies **min-max fair share** (mFS) if every agent gets at least her min-max fair share.

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An allocation $\overrightarrow{\pi}$ satisfies **min-max fair share** (mFS) if every agent gets at least her min-max fair share.

- mFS = the worst share an agent can get in a *"Someone cuts, I choose first"* game.
- In the **cake-cutting** case, same as PFS.

Min-max fair share: properties

Facts:

- Computing u_i^{mFS} for a given agent is hard $\rightarrow$ **coNP**-complete [PARTITION]
- Hence, deciding whether an allocation satisfies mFS is probably also hard (**NP**-complete?).
- $\overrightarrow{\pi}$ satisfies mFS $\Rightarrow$ $\overrightarrow{\pi}$ satisfies PFS.
- $\overrightarrow{\pi}$ is envy-free $\Rightarrow$ $\overrightarrow{\pi}$ satisfies mFS.

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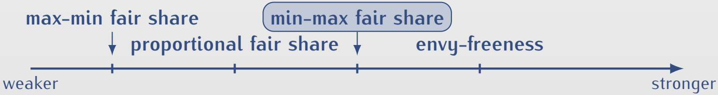

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For €1, what would you buy?

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- **Agent 1:** 1 and 4;
- **Agent 2:** 2 and 3.

Disjoint shares!

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- **Agent 2:** 2 and 3.

Disjoint shares!

$\Rightarrow$ Allocation $\langle\{1, 4\}, \{2, 3\}\rangle$, with prices $\langle 0.8, 0.2, 0.8, 0.2\rangle$ forms a CEEI.

$\Rightarrow$ Allocation $\langle\{1, 4\}, \{2, 3\}\rangle$ satisfies CEEI.

Competitive Equilibrium from Equal Incomes

- A classical notion in economics [Moulin, 1995]
- Subcase (indivisible goods) of the **Fisher model** [Walras, 1874, Fisher, 1892]
- Introduced recently in computer science [Othman et al., 2010]

Fisher, I. (1892).
Mathematical Investigations in the Theory of Value and Prices, and Appreciation and Interest.
Augustus M. Kelley, Publishers.

Moulin, H. (1995).
Cooperative Microeconomics, A Game-Theoretic Introduction.
Prentice Hall.

Othman, A., Sandholm, T., and Budish, E. (2010).
Finding approximate competitive equilibria: efficient and fair course allocation.
In *Proceedings of AAMAS'10*.

Walras, L. (1874).
Éléments d'économie politique pure ou Théorie de la richesse sociale.
L. Corbaz, 1 edition.

CEEI: known facts

Fact: $\vec{\pi}$ satisfies CEEI $\Rightarrow$ $\vec{\pi}$ is envy-free.

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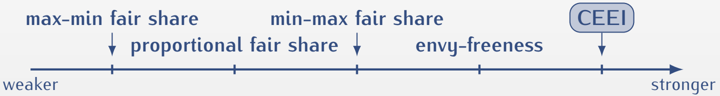

Fisher model: an equilibrium always exists – Nash ($\times$) optimal

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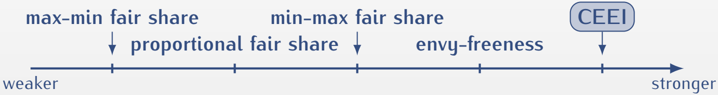

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Unfortunately, in the **discrete setting**, a CEEI may not exist.

Worse [Brânzei et al., 2015]...

- Is $(\vec{\pi}, \vec{p})$ a CEEI? $\rightarrow$ **coNP**-complete
- Does there exist a CEEI? **NP**-hard

Brânzei, S., Hosseini, H., and Miltersen, P. B. (2015).
Characterization and computation of equilibria for indivisible goods.
In *Algorithmic Game Theory*, pages 244–255. Springer.

CEEI: open problems

- Does there exist a CEEI? **NP**-hard and in Σ_2^P. Precise complexity?
- How to test whether $\vec{\pi}$ is a CEEI (and find the associated $\vec{p}$)?

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$$0 \leq \mathbf{p_o} \leq 1, \text{ for all } o \in [\![1, m]\!] \tag{1}$$

$$\sum_{o=1}^{m} a_{\pi_i,o}\mathbf{p_o} \leq 1, \text{ for all } i \in [\![1, n]\!], \text{with } a_{\pi_i,o} = 1 \text{ if } o \in \pi_i, \text{ 0 otherwise} \tag{2}$$

$$\sum_{o=1}^{m} a_{\pi',o}\mathbf{p_o} > 1, \text{ for all } \pi' \text{ such that } \exists i \text{ such that } u_i(\pi') > u_i(\pi_i) \tag{3}$$

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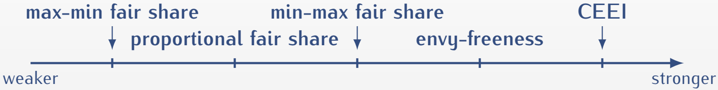
- How to compute a CEEI allocation?

Simplistic algorithm: compute all allocations and test which ones are CEEI.

Five fairness criteria

Summary and interpretation

Interpretation



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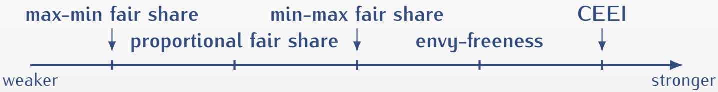

1 For all allocation $\overrightarrow{\pi}$:

$$(\overrightarrow{\pi} \models \text{CEEI}) \Rightarrow (\overrightarrow{\pi} \models \text{EF}) \Rightarrow (\overrightarrow{\pi} \models \text{mFS}) \Rightarrow (\overrightarrow{\pi} \models \text{PFS}) \Rightarrow (\overrightarrow{\pi} \models \text{MFS})$$

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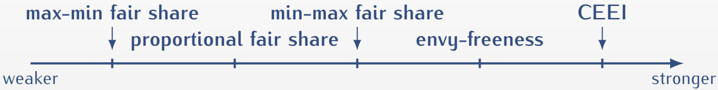

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2 If $\mathcal{I}_{|\mathcal{P}}$ is the set of instances s.t at least one allocation satisfies $\mathcal{P}$:

$$\mathcal{I}_{|\text{CEEI}} \subset \mathcal{I}_{|\text{EF}} \subset \mathcal{I}_{|\text{mFS}} \subset \mathcal{I}_{|\text{PFS}} \subset \mathcal{I}_{|\text{MFS}} \subset \mathcal{I}$$

$\rightarrow$ the lowest subset, the less “conflict-prone”.

Interpretation

weaker → stronger: max-min fair share, proportional fair share, min-max fair share, envy-freeness, CEEI

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Two extreme examples:

- 2 agents, 1 object $\rightarrow$ only in $\mathcal{I}_{|\text{MFS}}$
- 2 agents, 2 objects, with

	1	2
agent 1	1000	0
agent 2	0	1000

$\rightarrow$ in $\mathcal{I}_{|\text{CEEI}}$ (with *e.g.* $\overrightarrow{p} = \langle 1, 1 \rangle$).

A glimpse at experiments

What about fairness criteria in practice?

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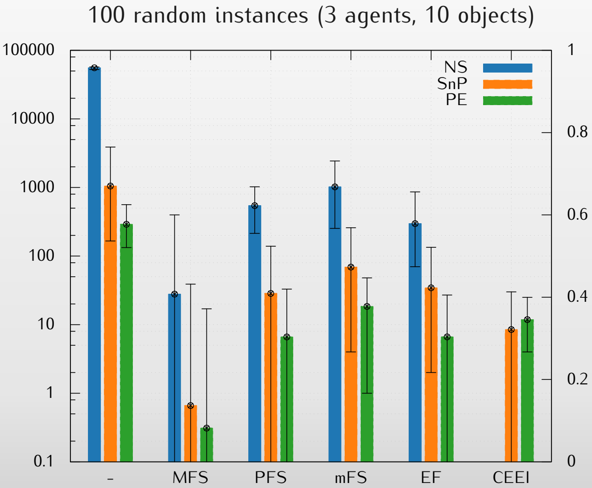
Goal of our experiments: evaluate the distribution of the allocation over:

- the fairness scale (–, MFS, PFS, mFS, EF, CEEI);
- three efficiency levels (–, Sequenceable, Pareto-efficient).

A glimpse at experiments



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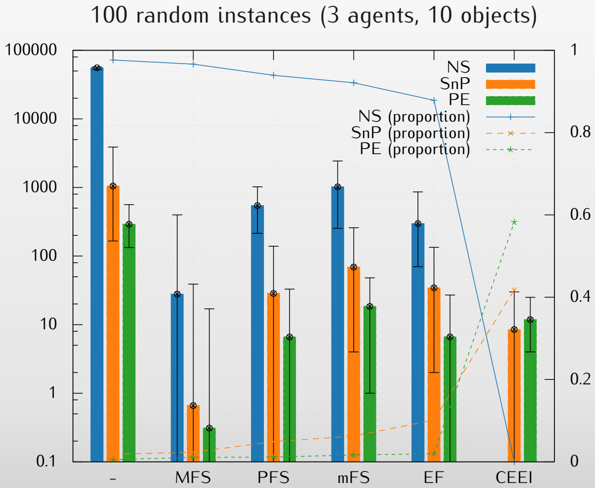

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A weight $w(\mathcal{S})$ to each subset $\mathcal{S}$ of objects (not only singletons) of size $\leq k$.
Note: additive $=$ 1-additive

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Examples:

- $w(skis) = 10$; $w(poles) = 0$; $w(\{skis, poles\}) = 90$
  $\rightarrow u(\{skis, poles\}) = 100 > 10 + 0$
- $w(skis) = 100$; $w(snowboard) = 100$; $w(\{skis, snowboard\}) = -100$
  $\rightarrow u(\{skis, snowboard\}) = 100 < 100 + 100$

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Reminder

For **additive preferences** we can almost always find an allocation satisfying max-min fair share.

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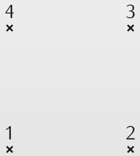
Example: 4 objects, 2 agents



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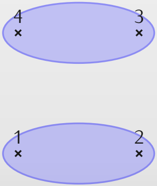

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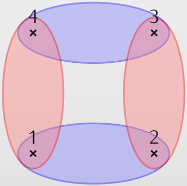

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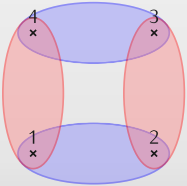

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Worse... Deciding whether there exists one is **NP**-complete [PARTITION].

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A possible approach to fairness in multiagent resource allocation problems:

1. Determine the highest satisfiable criterion.
2. Find an allocation that satisfies this criterion.
3. Explain to the upset agents that we cannot do much better.

What else

Some future directions...

- Link with a scale of efficiency criteria (recent work)
- Some missing complexity results
- Develop efficient **algorithms**
- More **experiments**
- Extend to **more expressive preference languages** (including ordinal ones...)