Picking Sequences for Resource Allocation

Sylvain Bouveret
LIG (STeamer) – Ensimag / Grenoble INP
Jérôme Lang
LAMSADE – Université Paris Dauphine

Séminaire de l'équipe POLARIS

Grenoble, March 10, 2016

A fair division problem. . .

You have. . .

- p candies of different kinds and flavors;
- n kids: kid A, kid B, kid C. . .

A fair division problem…

You have…

- p candies of different kinds and flavors;
- n kids: kid A, kid B, kid C…

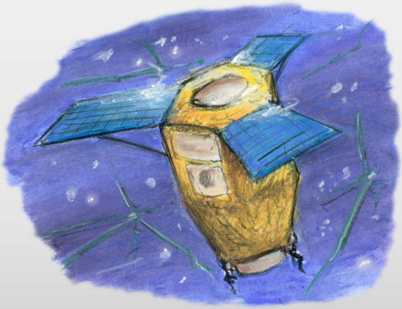
*How would you allocate the candies to the kids so as to be as **fair** as possible ?*

An ubiquitous problem

- Allocation of courses (or practical works) to students
- Allocation of take-off and landing slots in airports
- Allocation of tasks to workers
- Allocation of jobs to machines
- Allocation satellite resources

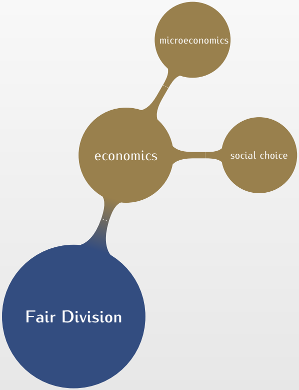
An ubiquitous problem

- Allocation of courses (or practical works) to students
- Allocation of take-off and landing slots in airports
- Allocation of tasks to workers
- Allocation of jobs to machines
- Allocation satellite resources

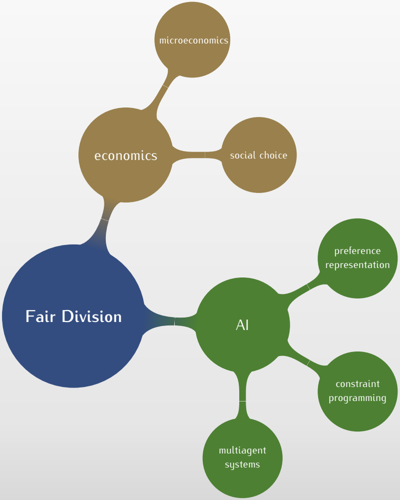


A rich problem

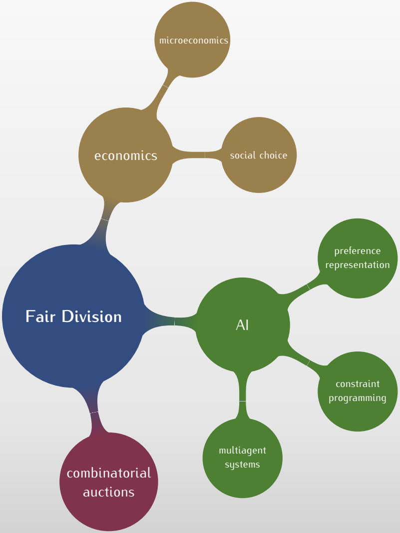
A rich problem



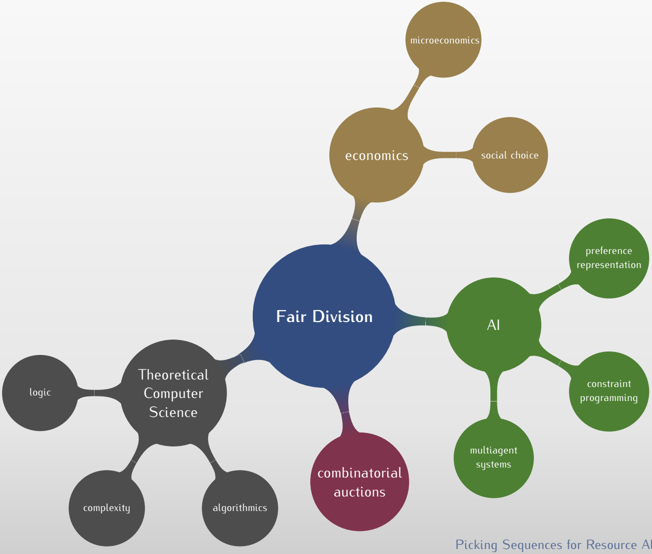
A rich problem



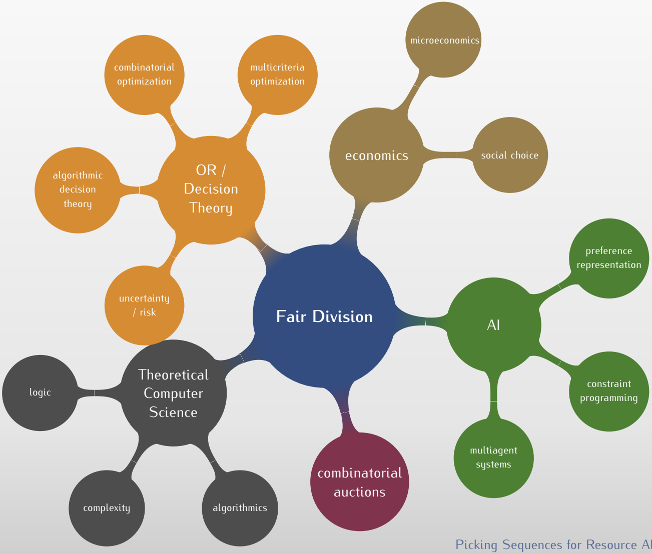
A rich problem



A rich problem



A rich problem



Computational social choice

Computational Social Choice (COMSOC)

COMSOC $\approx$ Social Choice $\cap$ Computer Science

Computational social choice

Computational Social Choice (COMSOC)

COMSOC $\approx$ Social Choice $\cap$ Computer Science

1. Use techniques from economics to solve problems in IT (network sharing, job allocation...)
2. Use techniques from CS to analyze and solve economical problems (complexity of voting procedures, compact preference representation...)

Computational social choice

Computational Social Choice (COMSOC)

COMSOC ≈ Social Choice ∩ Computer Science

1. Use techniques from economics to solve problems in IT (network sharing, job allocation...)
2. Use techniques from CS to analyze and solve economical problems (complexity of voting procedures, compact preference representation...)

- Fair division (indivisible goods or cake-cutting)
- Voting
- Coalition formation / hedonic games
- Judgment aggregation...

Back to our fair division problem

You have...

- p candies of different kinds and flavors;
- n kids: kid A, kid B, kid C...

Back to our fair division problem

You have...

- p candies of different kinds and flavors;
- n kids: kid A, kid B, kid C...

*How would you allocate the candies to the kids so as to be as **fair** as possible ?*

Back to our fair division problem

You have...

- p candies of different kinds and flavors;
- n kids: kid A, kid B, kid C...

How would you allocate the candies to the kids so as to be as ***fair*** *as possible ?*

- **Answer 1** (the social choice theoretician's): Ask the kids to give their preferences and use a (centralized) collective decision making procedure.
- **Answer 2** (the MAS specialist's): Ask the kids to negotiate.

Back to our fair division problem

You have…

- p candies of different kinds and flavors;
- n kids: kid A, kid B, kid C…

How would you allocate the candies to the kids so as to be as ***fair*** *as possible ?*

- **Answer 1** (the social choice theoretician's): Ask the kids to give their preferences and use a (centralized) collective decision making procedure.
- **Answer 2** (the MAS specialist's): Ask the kids to negotiate.

A much simpler procedure…

A much simpler procedure...

Ask the kids to pick in turn their most preferred candy among the remaining ones, according to some **predefined sequence**.

Example

3 kids A, B, C, 6 candies, sequence $ABCCBA \rightarrow A$ chooses first (and takes her preferred candy), then B, then C, then C again...

A much simpler procedure…

Ask the kids to pick in turn their most preferred candy among the remaining ones, according to some **predefined sequence**.

Example

3 kids A, B, C, 6 candies, sequence $ABCCBA \rightarrow A$ chooses first (and takes her preferred candy), then B, then C, then C again…

We "feel" that $ABCCBA$ is **fairer** than $AABBCC$…

$\rightarrow$ *What is the **fairest** sequence ?*

The model

What is the best sequence?

Agents, objects, preferences...

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A central authority (CA) must find a **policy** (a sequence of agents) $\pi : \{1, \ldots, p\} \rightarrow \{A, B, \ldots, x\}$
- The central authority assumes that each agent i has a (private) **ranking** $\succ_i$ over $\mathcal{O}$ (ex: $6 \succ 1 \succ 4 \succ 5 \succ 2 \succ 3$)

Example

Example

5 objects, 3 agents, $\pi = ABCCB\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

Example

Example

5 objects, 3 agents, $\pi = ABCCB\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

k	0
$s(A)^\pi_k$	$\emptyset$
$s(B)^\pi_k$	$\emptyset$
$s(C)^\pi_k$	$\emptyset$
$\mathcal{O}^\pi_k$	$\emptyset$

Example

Example

5 objects, 3 agents, $\pi = \mathbf{A}BCCB\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

k	0	1
$s(A)_k^\pi$	$\emptyset$	1
$s(B)_k^\pi$	$\emptyset$	$\emptyset$
$s(C)_k^\pi$	$\emptyset$	$\emptyset$
$\mathcal{O}_k^\pi$	$\emptyset$	1

Example

Example

5 objects, 3 agents, $\pi = A\mathbf{B}CCB\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

k	0	1	2
$s(A)_k^\pi$	$\emptyset$	1	1
$s(B)_k^\pi$	$\emptyset$	$\emptyset$	4
$s(C)_k^\pi$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathcal{O}_k^\pi$	$\emptyset$	1	14

Example

Example

5 objects, 3 agents, $\pi = AB\mathbf{C}CB\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

k	0	1	2	3
$s(A)^\pi_k$	$\emptyset$	1	1	1
$s(B)^\pi_k$	$\emptyset$	$\emptyset$	4	4
$s(C)^\pi_k$	$\emptyset$	$\emptyset$	$\emptyset$	3
$\mathcal{O}^\pi_k$	$\emptyset$	1	14	143

Example

Example

5 objects, 3 agents, $\pi = ABC\mathbf{C}B\ldots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

k	0	1	2	3	4
$s(A)^{\pi}_{k}$	$\emptyset$	1	1	1	1
$s(B)^{\pi}_{k}$	$\emptyset$	$\emptyset$	4	4	4
$s(C)^{\pi}_{k}$	$\emptyset$	$\emptyset$	$\emptyset$	3	35
$\mathcal{O}^{\pi}_{k}$	$\emptyset$	1	14	143	1435

Example

Example

5 objects, 3 agents, $\pi = ABCC\mathbf{B}\dots$

- $A : 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $B : 4 \succ 2 \succ 5 \succ 1 \succ 3$
- $C : 1 \succ 3 \succ 5 \succ 4 \succ 2$

k	0	1	2	3	4	5
$s(A)_k^\pi$	$\emptyset$	1	1	1	1	1
$s(C)_k^\pi$	$\emptyset$	$\emptyset$	4	4	4	42
$s(C)_k^\pi$	$\emptyset$	$\emptyset$	$\emptyset$	3	35	35
$\mathcal{O}_k^\pi$	$\emptyset$	1	14	143	1435	14352

Scoring functions

We only have rankings over objects...

$\rightarrow$ *How to compare two allocations ?*

Scoring functions

We only have rankings over objects...

$\rightarrow$ *How to compare two allocations ?*

Two natural assumptions:

1. **Scoring:** We have a common **scoring function** $g : \{1, \ldots, p\} \mapsto \mathbb{N}$ mapping each rank to a utility.
2. **Additivity:** These utilities are **additive**.

Scoring functions

We only have rankings over objects...

→ *How to compare two allocations ?*

Two natural assumptions:

1. **Scoring:** We have a common **scoring function** $g : \{1, \dots, p\} \mapsto \mathbb{N}$ mapping each rank to a utility.
2. **Additivity:** These utilities are **additive**.

3 natural scoring functions:

$\succ_i$	6	1	4	5	2	3

Scoring functions

We only have rankings over objects…

→ *How to compare two allocations ?*

Two natural assumptions:

1. **Scoring:** We have a common **scoring function** $g : \{1, \ldots, p\} \mapsto \mathbb{N}$ mapping each rank to a utility.
2. **Additivity:** These utilities are **additive**.

3 natural scoring functions:

$\succ_i$	6	1	4	5	2	3
Borda	6	5	4	3	2	1

Scoring functions

We only have rankings over objects...

→ *How to compare two allocations ?*

Two natural assumptions:

1. **Scoring:** We have a common **scoring function** $g : \{1, \ldots, p\} \mapsto \mathbb{N}$ mapping each rank to a utility.
2. **Additivity:** These utilities are **additive**.

3 natural scoring functions:

$\succ_i$	6	1	4	5	2	3
Borda	6	5	4	3	2	1
lexicographic	32	16	8	4	2	1

Scoring functions

We only have rankings over objects...

→ *How to compare two allocations ?*

Two natural assumptions:

1. **Scoring:** We have a common **scoring function** $g : \{1, \dots, p\} \mapsto \mathbb{N}$ mapping each rank to a utility.
2. **Additivity:** These utilities are **additive**.

3 natural scoring functions:

$\succ_i$	6	1	4	5	2	3
Borda	6	5	4	3	2	1
lexicographic	32	16	8	4	2	1
Quasi-Indifference	$1+5\varepsilon$	$1+4\varepsilon$	$1+3\varepsilon$	$1+2\varepsilon$	$1+\varepsilon$	1

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB\ldots$

- $A : \mathbf{1} \succ 2 \succ 3 \succ 4 \succ 5$
- $B : \mathbf{4} \succ \mathbf{2} \succ 5 \succ 1 \succ 3$
- $C : 1 \succ \mathbf{3} \succ \mathbf{5} \succ 4 \succ 2$

With π, agent A gets 1, agent B gets 24, agent C gets 35

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB\ldots$

- $A : \mathbf{1} \succ 2 \succ 3 \succ 4 \succ 5$
- $B : \mathbf{4} \succ \mathbf{2} \succ 5 \succ 1 \succ 3$
- $C : 1 \succ \mathbf{3} \succ \mathbf{5} \succ 4 \succ 2$

With π, agent A gets 1, agent B gets 24, agent C gets 35

- **Borda:** $u_A(\pi) = 5$; $u_B(\pi) = 5 + 4 = 9$; $u_C(\pi) = 4 + 3 = 7$.
- **lexicographic:** $u_A(\pi) = 16$; $u_B(\pi) = 24$; $u_C(\pi) = 12$.
- **QI:** $u_A(\pi) = 1 + 4\varepsilon$; $u_B(\pi) = 2 + 7\varepsilon$; $u_C(\pi) = 2 + 5\varepsilon$.

Social welfare

We use a **collective utility function** to aggregate the individual utilities.

Two well-known functions:

- **utilitarian:** $F(u_A, \ldots, u_x) = \sum_{i \in \mathcal{N}} u_i$;
- **egalitarian:** $F(u_A, \ldots, u_x) = \min_{i \in \mathcal{N}} u_i$.

Uncertainty

The procedure is elicitation-free...

$\rightarrow$ *Which information can we use to find the best sequence ?*

Uncertainty

The procedure is elicitation-free...

$\rightarrow$ *Which information can we use to find the best sequence ?*

The CA has a **probabilistic model** of the preferences:

- **Full independence** : each profile $R = \langle \succ_A, \ldots, \succ_x \rangle$ is equally probable
- **Full correlation** : all the agents have the same ranking $(R = \langle \succ, \ldots, \succ \rangle)$

Uncertainty

The procedure is elicitation-free...

→ *Which information can we use to find the best sequence ?*

The CA has a **probabilistic model** of the preferences:

- **Full independence** : each profile $R = \langle \succ_A, \dots, \succ_x \rangle$ is equally probable
- **Full correlation** : all the agents have the same ranking ($R = \langle \succ, \dots, \succ \rangle$)

Expected individual and collective utilities:

$$\overline{u(i,\pi)} = \sum_{R \in Prof(\mathcal{N},\mathcal{O})} Pr(R) \times u_i(\pi, R).$$

$$\overline{sw_F(\pi)} = F(\overline{u(1,\pi)}, \dots, \overline{u(n,\pi)}).$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.
What is agent 3's expected utility with this sequence ?

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.
What is agent 3's expected utility with this sequence ?
3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3, \pi)} =$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.
What is agent 3's expected utility with this sequence ?
3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3,\pi)} = \frac{1}{\binom{5}{2}} \times (3+2)$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.

What is agent 3's expected utility with this sequence ?

3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3,\pi)} = 0.5 + \frac{1}{\binom{5}{2}} \times (4+2)$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.

What is agent 3's expected utility with this sequence ?

3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3, \pi)} = 0.5 + 0.6 + \frac{1}{\binom{5}{2}} \times (5 + 2)$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.

What is agent 3's expected utility with this sequence ?

3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3,\pi)} = 0.5 + 0.6 + 0.6 + \frac{2}{\binom{5}{2}} \times (4+3)$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.

What is agent 3's expected utility with this sequence ?

3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3,\pi)} = 0.5 + 0.6 + 0.6 + 1.4 + \frac{2}{\binom{5}{2}} \times (5+3)$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.

What is agent 3's expected utility with this sequence ?

3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3,\pi)} = 0.5 + 0.6 + 0.6 + 1.4 + 1.6 + \frac{\binom{3}{2}}{\binom{5}{2}} \times (5+4)$$

Back to the example

Example

5 objects, 3 agents, $\pi = ABCCB$, $g = g_{Borda}$, full independence.
What is agent 3's expected utility with this sequence ?
3's preferences: $? \succ ? \succ ? \succ ? \succ ?$

$$\overline{u(3,\pi)} = 0.5 + 0.6 + 0.6 + 1.4 + 1.6 + 2.7 = \mathbf{7.5}$$

Summary

- **Instance:**
  - a number of agents n
  - a number of objects p
  - a scoring function g
  - a correlation profile $Corr \in \{FC, FI\}$
  - a collective utility function F
- **Question:**
  - *What is the policy* π *maximizing* $\overline{sw_F(\pi)}$, *under correlation profile* $Corr$ *?*

Some general results

1. Full correlation

Some general results

1. Full correlation

Utilitarian CUF (sum)

All policies have the same expected value!

Some general results

1. Full correlation

Utilitarian CUF (sum)

All policies have the same expected value!

Egalitarian CUF (min)

Sequential allocation is NP-complete.

(Reduction from [PARTITION])

Some general results

1. Full correlation

Utilitarian CUF (sum)

All policies have the same expected value!

Egalitarian CUF (min)

Sequential allocation is **NP**-complete.

(Reduction from [PARTITION])

What about...

- ...lexicographic scoring ?
- ...quasi-indifference scoring ?
- ...Borda scoring ?

Lexicographic scoring

$\succ_i$	6	$\succ$	1	$\succ$	4	$\succ$	5	$\succ$	2	$\succ$	3
lexicographic	32	$>>$	16	$>>$	8	$>>$	4	$>>$	2	$>>$	1

Egalitarian CUF (min)

Optimal policies: $\sigma(A)\sigma(B)\ldots\sigma(x)\sigma(x)^{p-n}$ (where σ is a permutation of $\{A, B, \ldots, x\}$)

Lexicographic scoring

$\succ_i$	6	$\succ$	1	$\succ$	4	$\succ$	5	$\succ$	2	$\succ$	3
lexicographic	32	$>>$	16	$>>$	8	$>>$	4	$>>$	2	$>>$	1

Egalitarian CUF (min)

Optimal policies: $\sigma(A)\sigma(B)\ldots\sigma(x)\sigma(x)^{p-n}$ (where σ is a permutation of $\{A, B, \ldots, x\}$)

Example: $\pi = ABCCCC$

- $\overline{u(1,\pi)} = 32$
- $\overline{u(2,\pi)} = 16$
- $\overline{u(3,\pi)} = 8 + 4 + 2 + 1 = 15$

Borda scoring

$\succ_i$	6	1	4	5	2	3
Borda	6	5	4	3	2	1

Borda scoring

$\succ_i$	6	1	4	5	2	3
Borda	6	5	4	3	2	1

This is **polynomial** in p (dynamic programming algorithm).

QI scoring

$\succ_i$	6	1	4	5	2	3
Quasi-Indifference	$1+5\varepsilon$	$1+4\varepsilon$	$1+3\varepsilon$	$1+2\varepsilon$	$1+\varepsilon$	1

Egalitarian CUF (min)

Comes down to solving the Borda case!

QI scoring

$\succ_i$	6	1	4	5	2	3
Quasi-Indifference	$1+5\varepsilon$	$1+4\varepsilon$	$1+3\varepsilon$	$1+2\varepsilon$	$1+\varepsilon$	1

Egalitarian CUF (min)

Comes down to solving the Borda case!

Intuition:

- let $m = \lfloor \frac{p}{n} \rfloor$ and $q = p - nm$
- Optimal policies: $\pi = \underbrace{AABB}_{n-q \text{ agents}} \overbrace{CCCDDD}^{q \text{ agents}}$ and $\pi' = \underbrace{ABBA}_{n-q \text{ agents}} \overbrace{CCCDDD}^{q \text{ agents}}$
- The q last agents are OK $\rightarrow u \geq m+1$
- The $n-q$ first agents: $u = m + x \cdot \varepsilon$ ($x \rightarrow$ Borda)

A complex problem. . .

2. Full independence

A complex problem...

2. **Full independence**

Conjecture (2011)

Computing the expected utility of a sequence is NP-complete.
Computing the **optimal** sequence probably harder.

Results

Computing the expected utility of a sequence is **NP**-complete.

Results

Computing the expected utility of a sequence is ~~NP-complete~~ **polynomial** [Kalinowski et al., 2013].

Kalinowski, T., Narodytska, N., and Walsh, T. (2013).
A social welfare optimal sequential allocation procedure.
In *Proceedings of IJCAI 2013.*

Results

Computing the expected utility of a sequence is ~~NP-complete~~ **polynomial** [Kalinowski et al., 2013].

Computing the **optimal** sequence probably harder.

Kalinowski, T., Narodytska, N., and Walsh, T. (2013).
A social welfare optimal sequential allocation procedure.
In *Proceedings of IJCAI 2013.*

Results

Computing the expected utility of a sequence is ~~NP-complete~~ **polynomial** [Kalinowski et al., 2013].

Computing the **optimal** sequence probably harder.

- the **alternating policy** (*ABABABAB*...) is optimal for Borda, utilitarian social welfare
- complexity unknown for other social welfare and scoring functions (NP-hardness conjectured)

Kalinowski, T., Narodytska, N., and Walsh, T. (2013).
A social welfare optimal sequential allocation procedure.
In *Proceedings of IJCAI 2013.*

Some examples

Assumptions: Full independence, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4		
5		
6		
8		
10		

Some examples

Assumptions: Full independence, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4	ABBA	
5		
6		
8		
10		

Some examples

Assumptions: Full independence, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4	ABBA	ABCC
5		
6		
8		
10		

Some examples

Assumptions: Full independence, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4	ABBA	ABCC
5	AABBB	ABCCB
6		
8		
10		

Some examples

Assumptions: Full independence, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4	ABBA	ABCC
5	AABBB	ABCCB
6	ABABBA	ABCCBA
8	ABBABAAB	AACCBBCB
10	ABBAABABBA	ABCABBCACC

Some examples

Assumptions: Full independence, egalitarian CUF, Borda scoring function.

p	$n = 2$	$n = 3$
4	ABBA	ABCC
5	AABBB	ABCCB
6	ABABBA	ABCCBA
8	ABBABAAB	AACCBBCB
10	ABBAABABBA	ABCABBCACC

Other examples on
`http://recherche.noiraudes.net/en/sequences.php`

Strategical issues (manipulation)

Is the protocol strategy-proof?

Manipulation?

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A central authority (CA) has chosen a **policy** π and will execute it
- The agents have their own private preferences $\rightarrow$ **picking strategy**.

Manipulation?

Example

2 agents, 4 objects:

- A: $1 \succ 2 \succ 3 \succ 4$
- B: $2 \succ 3 \succ 4 \succ 1$

Sequence $\pi = ABBA \rightarrow \{14|23\}$.

Manipulation?

Example

2 agents, 4 objects:

- A: $1 \succ 2 \succ 3 \succ 4$
- B: $2 \succ 3 \succ 4 \succ 1$

Sequence $\pi = ABBA \rightarrow \{14|23\}$.

What if A knows B's preferences and acts maliciously?

Manipulation?

Example

2 agents, 4 objects:

- A: $1 \succ 2 \succ 3 \succ 4$
- B: $2 \succ 3 \succ 4 \succ 1$

Sequence $\pi = ABBA \rightarrow \{14|23\}$.

What if A knows B's preferences and acts maliciously?

She can manipulate by picking 2 instead of 1 at first step $\rightarrow \{12|34\}$.

More formally

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A **policy** π
- The agents have their own private preferences (which may or may not be additive) and use them for their **picking strategy**.

More formally

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A **policy** π
- The agents have their own private preferences (which may or may not be additive) and use them for their **picking strategy**.

The cheating agent (A) knows:

- the sequence
- her own (general) **picking strategy**
- the others' **picking strategy** (assumed to be **simple** and **deterministic** – as if each agent had an underlying linear order over the objects)

More formally

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A **policy** π
- The agents have their own private preferences (which may or may not be additive) and use them for their **picking strategy**.

The cheating agent (A) knows:

- the sequence
- her own (general) **picking strategy**
- the others' **picking strategy** (assumed to be **simple** and **deterministic** – as if each agent had an underlying linear order over the objects)

She wants:

- to get the best bundle she can get.

More formally

- A set $\mathcal{O}$ of p objects $\{1, \ldots, p\}$
- A set $\mathcal{N}$ of n agents $\{A, B, \ldots, x\}$
- A **policy** π
- The agents have their own private preferences (which may or may not be additive) and use them for their **picking strategy**.

The cheating agent (A) knows:

- the sequence
- her own (general) **picking strategy**
- the others' **picking strategy** (assumed to be **simple** and **deterministic** – as if each agent had an underlying linear order over the objects)

She wants:

- to get the best bundle she can get.

Her only possible cheating actions:

- choose at given steps **not** to pick her preferred objects.

First result

A: *"Can I get $\mathcal{S}$ for sure?"*

First result

A: *"Can I get $\mathcal{S}$ for sure?"*

Getting a subset for sure

We can answer to that constructively in polynomial time!

First result

A: *"Can I get $\mathcal{S}$ for sure?"*

Getting a subset for sure

We can answer to that constructively in polynomial time!

Idea:

- two agents: pick the objects in $\mathcal{S}$ in reverse order of $\succ_B$
- more agents:
  - transform agents 2 to $m-1$ into a single (fake) agent
  - apply the algorithm for 2 agents

General manipulation problem

A: *"What is the best subset I can get?"*

General manipulation problem

A: *"What is the best subset I can get?"*

Idea: Greedily build the optimal achievable subset:

- Find the best object i such that $\{i\}$ is achievable;
- Find the best object j such that $\{i, j\}$ is achievable;
- ...

General manipulation problem

A: *"What is the best subset I can get?"*

Idea: Greedily build the optimal achievable subset:

- Find the best object i such that $\{i\}$ is achievable;
- Find the best object j such that $\{i, j\}$ is achievable;
- ...

Manipulation with additive preferences, two agents

If the manipulator has additive preferences, the optimal manipulation can be computed in polynomial time.

General manipulation problem

A: *"What is the best subset I can get?"*

Idea: Greedily build the optimal achievable subset:

- Find the best object i such that $\{i\}$ is achievable;
- Find the best object j such that $\{i, j\}$ is achievable;
- ...

Manipulation with additive preferences, two agents

If the manipulator has additive preferences, the optimal manipulation can be computed in polynomial time.

Only works **for two agents**!

General manipulation problem

Manipulation with additive preferences

If the manipulator has additive preferences, the optimal manipulation problem is NP-complete.

(reduction from [3-SAT])

Coalitional Manipulation

Example

3 agents, 6 objects:

- A: $1 \succ 2 \succ 5 \succ 4 \succ 3 \succ 6$
- B: $1 \succ 3 \succ 5 \succ 2 \succ 4 \succ 6$
- C: $2 \succ 3 \succ 4 \succ 1 \succ 5 \succ 6$

Sequence $\pi = ABCABC \rightarrow \{15|34|26\}$.

Coalitional Manipulation

Example

3 agents, 6 objects:

- A: $1 \succ 2 \succ 5 \succ 4 \succ 3 \succ 6$
- B: $1 \succ 3 \succ 5 \succ 2 \succ 4 \succ 6$
- C: $2 \succ 3 \succ 4 \succ 1 \succ 5 \succ 6$

Sequence $\pi = ABCABC \to \{15|34|26\}$.

- If A and B manipulate alone, they cannot do better
- If they cooperate, they can get $\{12|35|46\}$, which is strongly better.

Coalitional Manipulation: Results

Three kinds of manipulation considered here:

- No post-allocation trade allowed between the manipulators
- Post-allocation exchange of goods allowed between the manipulators
- Post-allocation exchange of goods + side-payments allowed

Coalitional Manipulation: Results

Three kinds of manipulation considered here:

- No post-allocation trade allowed between the manipulators
- Post-allocation exchange of goods allowed between the manipulators
- Post-allocation exchange of goods + side-payments allowed

Results:

- No post-allocation trade allowed between the manipulators → **NP**-complete [Partition]
- Post-allocation exchange of goods allowed between the manipulators → **NP**-complete [Partition]
- Post-allocation exchange of goods + side-payments allowed → polynomial (comes down to manipulation by a single agent)

Everyone manipulates...

One manipulator

Everyone manipulates...

One manipulator $\rightarrow$ several manipulators (coalitional manipulation)

Everyone manipulates...

One manipulator → several manipulators (coalitional manipulation)
→ everyone (rational, self-interested) manipulates?

Everyone manipulates...

One manipulator $\rightarrow$ several manipulators (coalitional manipulation)
$\rightarrow$ everyone (rational, self-interested) manipulates?

Game Theory (Subgame Perfect Nash Equilibrium)

Everyone manipulates...

One manipulator $\rightarrow$ several manipulators (coalitional manipulation)
$\rightarrow$ everyone (rational, self-interested) manipulates?

Game Theory (Subgame Perfect Nash Equilibrium)

- Two agents and additive utilities, precise characterization of the result of every SPNE $((rev(\succ_2), rev(\succ_1), rev(\pi)))$ [Kalinowski et al., 2013, Kohler and Chandrasekaran, 1971].
- Unbounded number of agents: **PSPACE**-hard [Kalinowski et al., 2013].

Kalinowski, T., Narodytska, N., Walsh, T., and Xia, L. (2013).
Strategic behavior when allocating indivisible goods sequentially.
In *Proceedings of AAAI'13*.

Kohler, D. A. and Chandrasekaran, R. (1971).
A class of sequential games.
Operations Research, 19(2):270–277.

Conclusion

A take-away message?

Conclusion

- A simple and intuitive sequential allocation procedure (actually already known in sparse litterature)
- Finding the best policy?
  - Full Correlation case well understood
  - Full Independence: partial results
- Strategical issues: individual and coalitional manipulation; game-theoretic approach

Sequenceability and Pareto-optimality

Another interesting property [Brams and King, 2005]:
Sequenceability exactly characterizes allocations that are ordinally necessary Pareto-optimal.

(an allocation can be generated by a sequence if and only if it is ordinally necessary Pareto-optimal)

Brams, S. and King, D. (2005).
Efficient fair division: Help the worst off or avoid envy ?
Rationality and Society, 17:387–421.